

ADMISSIBLE DECOMPOSITION OF COMPLEX PROJECTIVE STRUCTURES

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ABSTRACT. Let S be a closed orientable surface of genus at least two, and let C be an arbitrary (complex) projective structure on S . We show that there is a decomposition of S into pairs of pants and cylinders such that the restriction of C to each component has an injective developing map and a discrete and faithful holonomy representation. This decomposition implies that every projective structure can be obtained by the construction of Gallo, Kapovich, and Marden. Along the way, we show that there is an admissible loop on (S, C) , along which a grafting can be done.

1. INTRODUCTION

Let F be a connected orientable C^1 -smooth surface possibly with boundary, and let \tilde{F} denote the universal cover of F . A (*complex*) *projective structure* C on F is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2, \mathbb{C}))$ -structure, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. In other words, it is a maximal atlas of F modeled on $\hat{\mathbb{C}}$ with transition maps in $\mathrm{PSL}(2, \mathbb{C})$. The pair (F, C) is called a *projective surface*. As usual, we will often conflate C and (F, C) .

There is an equivalent definition, which we will mostly use in this paper: A projective structure is a pair (f, ρ) , where $f : \tilde{F} \rightarrow \hat{\mathbb{C}}$ is a C^1 -smooth locally injective map and $\rho : \pi_1(F) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is a homomorphism, such that f is ρ -equivariant, i.e. $f \circ \gamma = \rho(\gamma) \circ f$ for all $\gamma \in \pi_1(F)$. Then f is called the *developing map* and ρ the *holonomy (representation)* of the projective structure. On the interior of \tilde{F} , f is a local homeomorphism, and the restriction of f to each boundary component of \tilde{F} is a C^1 -smooth curve.

A projective structure $C = (f, \rho)$ is defined up to an isotopy of F and the action of an element of $\mathrm{PSL}(2, \mathbb{C})$, i.e. the post-composition of f with $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ and the conjugation of ρ by γ . (See [Th2, 3.4], [Ka2, 7.1].) The C^1 -smoothness is required to define a natural topology on the space of all projective structures on F in the case that F is not

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closed (see [CEG, I.1.5]). In this paper, we always assume that surfaces are connected and C^1 -smooth possibly with boundary (although we do not deform a projective structure).

Definition 1.1. *A projective structure $C = (f, \rho)$ on F is **admissible** if f is an embedding and ρ is an isomorphism onto a Schottky group or a quasifuchsian group. Analogously, a simple loop l on the projective surface (F, C) is **admissible** if \tilde{l} embeds to $\hat{\mathbb{C}}$ by f and $\rho(\gamma_l)$ is loxodromic, where \tilde{l} is a lift of l to \tilde{F} and γ_l is the homotopy class of l .*

A hyperbolic structure is a basic example of a projective structure, since $\mathbb{H}^2 \subset \hat{\mathbb{C}}$ and $\text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$ in a compatible way. Besides, every hyperbolic structure on a closed orientable surface is an admissible projective structure. However, in general, developing maps are not necessarily injective and holonomy representations are not necessarily discrete or faithful (c.f. (i) and (ii) below). Throughout this paper, let S denote a closed orientable surface of genus at least 2 and \tilde{S} denote the universal cover of S . (Orientability of S is not essential for the main theorems of this paper, if we consider two-dimensional Mobius structures instead of projective structures.) The following theorem yields a decomposition of an arbitrary projective surface into admissible projective subsurfaces:

Theorem 7.1 Let C be a projective structure on S . Then there exists a decomposition of S into cylinders and compact connected surfaces of negative Euler characteristic, such that the restriction of C to each cylinder is an integral flat structure and the restriction to each surface of negative Euler characteristic is an admissible projective structure.

An *integral flat structure* is a basic projective (actually affine) structure on a cylinder, which can be obtained by an operation called *grafting* (§3.2). If there is an admissible loop on a projective surface, we can define a grafting along this loop (see [Ka2], [Br] for example). This operation gives another projective structure on the same surface, preserving the orientation and the holonomy representation. If the admissible loop is circular, i.e. it corresponds to a simple circular arc on $\hat{\mathbb{C}}$ via the developing map, then the integral flat structure is exactly the structure that the grafting operation inserts to the projective structure along the loop. A flat structure on an annulus can be easily decomposed into admissible flat structures.

Theorem 7.1 immediately implies:

Corollary 7.2 Let C be an arbitrary projective structure on S . Then there exists a decomposition of S into pairs of pants and cylinders such that the restriction of C to each cylinder is an integral flat structure and the restriction to each pair of pants is an admissible structure.

This corollary gives the affirmative answer to a question raised by Gallo, Kapovich, and Marden ([GKM, 12.1]). The authors of [GKM] gave necessary and sufficient conditions for a representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ to be the holonomy representation of some projective structure on S . The conditions are: (i) $\mathrm{Im}(\rho)$ is a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and (ii) ρ lifts to a representation from $\pi_1(S)$ to $\mathrm{SL}(2, \mathbb{C})$. In order to prove the sufficiency, given an arbitrary representation ρ satisfying (i) and (ii), they constructed a projective structure on S with holonomy ρ in the following way: First, decompose S into pairs of pants, $\{P_i\}$, such that $\rho|_{\pi_1(P_i)}$ is an isomorphism onto a rank-two Schottky group for each i . Second, construct an admissible projective structure on each P_i with the holonomy representation $\rho|_{\pi_1(P_i)}$. Last, glue these structures on the pairs of pants together by inserting projective structures on cylinders between the corresponding boundaries of P_i 's, and obtain a desired projective structure. They asked whether every projective structure on S arises from such a Schottky pants decomposition. More specifically, they asked if every projective structure contains an admissible loop, which is answered by:

Theorem 6.2 For every projective structure C on S , there exists an admissible loop on (S, C) .

Theorem 6.2 is weaker than Theorem 7.1, since the boundary components of each subsurface in Theorem 7.1 are, in particular, admissible loops. However, the proof of Theorem 6.2 contains the basic ideas for the proof of Theorem 7.1. Furthermore, Theorem 6.2 addresses the following question about grafting:

Question 1 ([GKM, §12]): Assume that two projective structures on S have the same orientation and holonomy representation. Can one projective structure be transformed to the other by a sequence of graftings and inverse-graftings?

The grafting and inverse-grafting operations generate an equivalence relation among the projective structures with a given holonomy representation. Question 1 asks if there are exactly two equivalence classes represented by the orientations of the projective structures. Theorem

6.2 implies that every equivalence class consists of infinitely many projective structures. Ultimately, Question 1 aims to characterize the collection of projective structures with the given holonomy representation (see [Ka1], [GKM]). In the special case that the holonomy representation is an isomorphism onto a quasifuchsian group, the characterization is given by Goldman, using grafting, and the answer to Question 1 is affirmative ([Go1, Theorem C]). Also, we may compare this discussion about Question 1 with the fact that Dehn twists generate the mapping class group of S and yield a finite presentation of this group.

The *holonomy map* $Hol : P(S) \rightarrow V(S)$ is a projection given by $C = (f, \rho) \mapsto \rho$, where $P(S)$ is the space of all projective structures on S and $V(S)$ is the representation variety of homomorphisms from $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{C})$. This map is *not* a covering map onto its image ([He]), which makes problems in this area difficult.

One may ask the above questions in the case of other (G, X) -structures as well (c.f. [Go1, 1.10]). In particular, S. Choi gave a canonical decomposition of real projective structures, i.e. $(\mathrm{PGL}(3, \mathbb{R}), \mathbb{RP}^2)$ -structures, analogous to the one given by Theorem 7.1 (see [Ch]).

An outline of the proofs: For a given projective surface (S, C) , there is a corresponding pair $(\tau, L = (\lambda, \mu))$ of a marked hyperbolic structure τ on S and a measured geodesic lamination L on (S, τ) (see Thurston's coordinates in §3.8). A periodic leaf of λ corresponds to a continuous family of admissible loops on (S, C) . For each irrational minimal sublamination of λ , consider a standard sequence (l_i) of simple geodesic loops on (S, τ) that approximates λ (§3.5). Then we will show that l_i is admissible for all sufficiently large i . These admissible loops yield Theorem 6.2. Taking a disjoint union of such admissible loops, we construct a multi-loop on (S, C) well approximating the entire $|\lambda|$, the union of leaves of λ . We will show that the corresponding multi-loop on (S, C) achieves the desired admissible decomposition described in Theorem 7.1.

Every measured lamination on \mathbb{H}^2 induces a continuous map from \mathbb{H}^2 to \mathbb{H}^3 , called a *bending map* (§3.6). Via the bending map, the measure lamination corresponds to a projective structure on $\mathring{\mathbb{D}}^2$ (§3.8). Our proofs are based on the fact that injective quasiisometric (bending) maps correspond to admissible projective structures on $\mathring{\mathbb{D}}^2$ (§5). In order to show that l_i is admissible, we take the *total lift* \tilde{L} of L to \mathbb{H}^2 (§3.3) and a lift \tilde{l}_i of l_i to \mathbb{H}^2 . Consider the sublamination of \tilde{L} that consists of the leaves of \tilde{L} intersecting \tilde{l}_i , so that the structure on \tilde{l}_i embeds into the projective structure on $\mathring{\mathbb{D}}^2$ corresponding to the

sublamination. In other words, \tilde{L} and the sublamination coincide in a sufficiently small neighborhood of \tilde{l}_i , and therefore, the sublamination is sufficient to capture the structure on \tilde{l}_i and l_i . Thus sublamination induces an injective quasiisometric embedding since the “bending” is uniformly small (§4, §5). Then the sublamination of \tilde{L} corresponds to an admissible structure on $\mathring{\mathbb{D}}^2$. Since the structure on \tilde{l}_i is embedded in the admissible structure on $\mathring{\mathbb{D}}^2$, therefore l_i is also admissible. Theorem 7.1 will be proven based on the same idea.

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2. NOTATION

\overline{xy} : the geodesic segment connecting x and y , where x and y are points in a metric space.

$cl(X)$: the closure of X , where X is a subset of a topological space.

\mathring{X} : the interior of X .

$Conv(Y)$: the convex hull of Y , where Y is a subset of a hyperbolic space.

$D_r(x)$: the closed disk of radius r centered at x in a hyperbolic space.

3. PRELIMINARIES

3.1. Measured Laminations. (For details, see [PH], [CEG], [CB], [Ka2], [Th2].) Let F be a Riemannian surface with a constant curvature, possibly with geodesic boundary. A *geodesic lamination* on F is a collection of disjoint simple complete geodesics on F whose union is a closed subset of F . Each geodesic of a geodesic lamination is called a *leaf*. For a measured lamination λ on F , let $|\lambda| \subset F$ denote the union leaves of λ . If $|\lambda| = F$, then λ is called a *foliation*. A *measured lamination* is a pair $L = (\lambda, \mu)$, where λ is a geodesic lamination, and μ is a transversal measure of λ . Let a leaf of L refer to a leaf of λ and $|L|$ refer to $|\lambda|$. We always assume that λ is the support of μ , i.e. if l is a leaf of L , then $\mu(s) > 0$ for every geodesic segment s that transversally intersects l . The *weight* of a leaf l of L is $\inf\{\mu(s)\}$, where s varies over all geodesic segments that transversally intersect l , i.e. the atomic transversal measure of l . We denote the weight of l by $w(l)$. If l is an isolated leaf of L , then $w(l) = \mu(s)$ for every geodesic segment s on F that transversally intersects $|L|$ exactly once at a point on l .

By convention, if a geodesic segment s is contained in a leaf of λ , then $\mu(s) = 0$. In addition, letting x, y be the end points of s , if x or y is contained in a leaf with positive weight, then the weight does not contribute to the value of $\mu(s)$, so that $\mu(s) = \sup \mu(s')$ where s' varies over all geodesic segments strictly contained in s , i.e. $s' \subset s \setminus \{x, y\}$.

Recall that S is a closed orientable hyperbolic surface. A measured lamination on S is *minimal* if it does not contain any proper sublamination. Every measured lamination L on S uniquely decomposes into a finite number of disjoint minimal laminations of the following two types: a periodic leaf with positive weight (*periodic minimal lamination*), and a measured lamination consisting of uncountably many bi-infinite geodesics (*irrational minimal lamination*).

Let $M = (\nu, \omega)$ be an irrational minimal sublamination of L . Then, since each minimal sublamination of L forms a closed subset of S , $|\nu|$ is also an open subset of $|\lambda|$. Here are other properties of M : Each leaf of ν is a dense subset of $|\nu|$. If s is a geodesic segment on S transversally intersecting ν , then $s \cap |\nu|$ is a Cantor set i.e., a closed, perfect subset with empty interior ([PH, Corollary 1.7.6, 1.7.7]). Therefore, no leaf of M has a positive weight.

3.2. Flat Cylinders. For every $\theta \in (0, 2\pi)$ and distinct $z_1, z_2 \in \hat{\mathbb{C}}$, let R_θ be the open region in $\hat{\mathbb{C}}$ bounded by two simple circular arcs connecting z_1 and z_2 such that the inner angles at the vertices z_1 and z_2 are equal to θ . Since R_θ is embedded in $\hat{\mathbb{C}}$, it is equipped with a canonical projective structure (whose developing map is the identity map). We call the structure a *crescent* of angle θ ; its projective structure only depends on the choice of θ .

Let α be a hyperbolic element of $\mathrm{PSL}(2, \mathbb{C})$ that fixes z_1 and z_2 . Then $\langle \alpha \rangle$, the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by α , is an infinite cyclic group acting on R_θ freely and properly discontinuously. By quotienting R_θ by $\langle \alpha \rangle$, we obtain a projective structure on a cylinder. We call the structure a *flat structure* of height θ ; it only depends on θ and the translation length of α . A flat structure of height θ on a cylinder forms a projective surface, which we call a *flat cylinder* of height θ .

We shall define a crescent and a flat cylinder for arbitrary $\theta > 0$, generalizing those for $0 < \theta < 2\pi$. For arbitrary $\theta > 0$, let $R_\theta = (0, \theta) \times (0, \infty) \subset \mathbb{R}^2$. Define $f_\theta : (0, \theta) \times (0, \infty) \rightarrow \mathbb{C}$ by $f_\theta(x, y) = y \cos x + \sqrt{-1} y \sin x$ (i.e. the polar coordinates). Then f_θ defines a projective structure on $R_\theta \cong \mathring{\mathbb{D}}^2$, which is a crescent of angle θ . For $a > 0$, let T_a be an automorphism of \mathbb{R}^2 defined by $T_a(x, y) = (x, ay)$. Define a homomorphism $\rho : \langle T_a \rangle \cong \mathbb{Z} \rightarrow \mathrm{PSL}(2, \mathbb{C})$ by $\rho(T_a)(z) = az$ for all z . Then f_θ is ρ -equivariant. Quotienting R_θ by the action of

$\langle T_a \rangle$, we obtain a flat cylinder of height θ . A flat cylinder of height θ is *integral* if θ is a multiple of 2π . The multiplier is called the *degree* of the integral flat cylinder, so that, for all $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, the cardinality of $f_\theta^{-1}(z)$ equals to the degree.

Let $C = (f_\theta, \rho_{id})$ be the crescent of angle $\theta > 0$ given in the form above, where $\rho_{id} : \pi_1(R_\theta) \rightarrow \text{PSL}(2, \mathbb{C})$ is the trivial representation. For each $x \in (0, \theta)$, f_θ takes $x \times (0, \infty)$ to a straight line on \mathbb{C} connecting 0 and ∞ . The collection of these lines, $\{ \{x\} \times (0, \infty) \mid x \in (0, \theta) \}$, forms a foliation λ_C on R_θ , which we call the *canonical foliation* on C . We also can define the canonical transversal measure μ_C of λ_C by $\mu_C(\overline{P_1 P_2}) = |x_1 - x_2|$ for all $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in R_\theta$. Note that $\{x_1\} \times (0, \infty)$ and $\{x_2\} \times (0, \infty)$ bound a crescent of height $\mu_C(\overline{P_1 P_2})$ contained in C . Call (λ_C, μ_C) the *canonical measured foliation* on C .

For each $y \in (0, \infty)$, $(0, \theta) \times \{y\}$ is orthogonal to (each leaf of) λ_C in terms of the angles obtained by pulling back the conformal structure on $\hat{\mathbb{C}}$ via f_θ . Besides, f_θ takes $(0, \theta) \times \{y\}$ to a (not necessarily simple) circular arc on $\hat{\mathbb{C}}$. The collection of these orthogonal lines $\{(0, \theta) \times \{y\} \mid y \in (0, \infty)\}$ forms a foliation on R_θ , which is dual to λ_C . By identifying the points on each leaf of the dual foliation, R_θ projects to a line. Since (λ_C, μ_C) and its dual foliation are invariant under the action of $\langle T_a \rangle$, we obtain the *canonical foliation* and its dual foliation on the flat cylinder $C/\langle T_a \rangle$. Accordingly, the flat cylinder projects to a circle by identifying the points on each leaf of the dual foliation.

Since $f_\theta : (0, \theta) \times (0, \infty) \rightarrow \mathbb{C}$ continuously extends to $\{0, \theta\} \times (0, \infty)$, we can compactify a flat cylinder of height θ to a projective structure on a compact cylinder with boundary. By abusing the notation, we call this compactified flat cylinder, also, a flat cylinder of angle θ . Accordingly, the universal cover of the compactified flat cylinder of angle θ also is called a crescent of angle θ .

3.3. Dual Tree. (For more details, see [Mo], [Ka2].) Let S be a closed hyperbolic surface and $L = (\lambda, \mu)$ be a measured lamination on S without periodic leaves. Then no leaf of L has a positive weight. Let $p : \mathbb{H}^2 \rightarrow S$ be the covering map. The *total lift* of L is a measured lamination $\tilde{L} = (\tilde{\lambda}, \tilde{\mu})$ on \mathbb{H}^2 , where $\tilde{\lambda}$ consists of all the lifts of the leaves of λ , and $\tilde{\mu}$ is the pull back of μ , so that $(\mathbb{H}^2, \tilde{L})$ is locally isomorphic to (S, L) via p . Then \tilde{L} is a $\pi_1(S)$ -invariant measured lamination on \mathbb{H}^2 .

There is a unique \mathbb{R} -tree dual to \tilde{L} constructed in the following way (if L contains periodic leaves, the construction is more complicated). The transversal measure $\tilde{\mu}$ defines a pseudo-metric $d_{\tilde{\mu}}$ on \mathbb{H}^2 , by $d_{\tilde{\mu}}(x, y) = \tilde{\mu}(\overline{xy})$ for all $x, y \in \mathbb{H}^2$. Since no leaf of \tilde{L} has a positive weight,

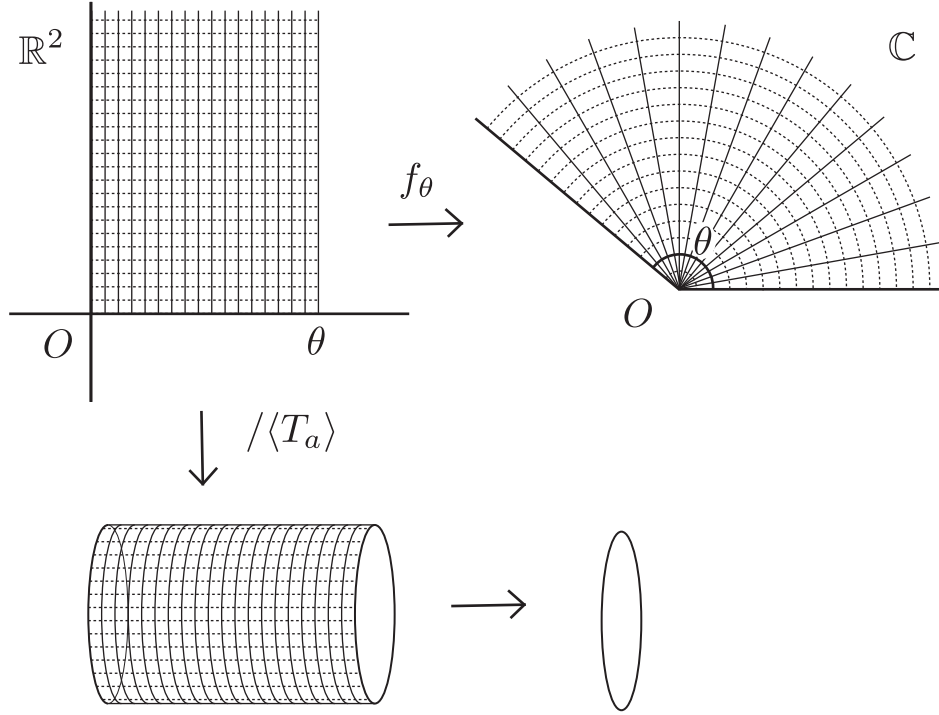


FIGURE 1.

$d_{\tilde{\mu}} : \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{R}_{\geq 0}$ is continuous. Define an equivalence relation on \mathbb{H}^2 by $x \sim y$ if and only if $\tilde{\mu}(\overline{xy}) = 0$. There are only two types of the equivalence classes: the closure of a complementary region of $|\tilde{\lambda}|$, and a leaf of $\tilde{\lambda}$ that is *not* a boundary geodesic of such a complementary region. Let T be the quotient of \mathbb{H}^2 by the equivalence relation, and let $\mathcal{P} : \mathbb{H}^2 \rightarrow T$ be the quotient map. In particular, if a geodesic in \mathbb{H}^2 is a leaf of \tilde{L} or it is contained in the closure of a complementary region of $|\tilde{\lambda}|$, then \mathcal{P} takes this geodesic to a point in T . It turns out that T is an \mathbb{R} -tree equipped with a canonical metric d_T induced by $\tilde{\mu}$: For every $x, y \in T$, $d_T(x, y) = \tilde{\mu}(\overline{x'y'})$, where $x', y' \in \mathbb{H}^2$ are such that $\mathcal{P}(x') = x$ and $\mathcal{P}(y') = y$. If a geodesic in \mathbb{H}^2 transversally intersects to \tilde{L} , then \mathcal{P} takes this geodesic to a geodesic in T . Since the action of $\pi_1(S)$ on \mathbb{H}^2 preserves \tilde{L} , $\pi_1(S)$ isometrically acts on T . Let l be a closed geodesic on S transversally intersecting L , \tilde{l} be a lift of l to \mathbb{H}^2 , and γ_l be the homotopy class of l in $\pi_1(S)$. Then $\mathcal{P}(\tilde{l})$ is a geodesic in T , and the action of γ_l on T isometrically translates along $\mathcal{P}(\tilde{l})$ by the distance $\mu(l)$.

3.4. Flow Boxes. Let (a, b) and (c, d) be open intervals in \mathbb{R} , and let Y be a closed subset of (c, d) . Consider a geodesic lamination λ on $(a, b) \times (c, d) \subset \mathbb{R}^2$ that consists of the leaves $(a, b) \times \{y\}$ for all $y \in Y$ (i.e. we have Y -worth of horizontal leaves). Let μ be a transversal measure for λ , and let L be the measured lamination (λ, μ) . The pair $((a, b) \times (c, d), L)$ is called a (*Euclidian*) *flow box*. Let s be a vertical geodesic in $(a, b) \times (c, d)$, i.e. $s = \{x\} \times (c, d)$ for some $x \in (a, b)$. The *height* of the flow box is $\mu(s)$, which does not depend on the choice of x . If L is a measured lamination on a hyperbolic quadrilateral Q , and (Q, L) is isomorphic to a flow box, then we also call (Q, L) a (hyperbolic) flow box.

Lemma 3.1. *Let $L = (\lambda, \mu)$ be a measured geodesic lamination on \mathbb{H}^2 without leaves of positive weight. Let s be a geodesic segment contained in a leaf of L . For every $\epsilon > 0$, there exists a neighborhood of s isomorphic to a flow box of height less than ϵ .*

Proof. Let l be the leaf of L containing s . Let s' be a geodesic segment strictly containing s , i.e. $s \subset s'$ and the end points of s and s' are all distinct. Let a and b be the geodesic segments orthogonal to s' and passing through the end points of s' . Let H_1 and H_2 be the components of $\mathbb{H}^2 \setminus l$, which are the open half planes bounded by l . Choose a geodesic l_i in H_i sufficiently close to l such that l_i does not intersect L transversally. For $i = 1, 2$, let R_i denote the open hyperbolic quadrilateral bounded by l , a , l_i , b . Let a_i and b_i be the opposite edges of R_i that are contained in a and b , respectively. Note that the edge of R_i contained in l is s' .

First, suppose that the leaves of L contained in H_i do *not* accumulate to l . If we choose l_i sufficiently close to l , R_i does not intersect leaves of L . Then R_i is a flow box with the empty lamination. Next, suppose that the leaves of L contained in H_i accumulate to l . If l_i is sufficiently close to l , then, for every leaf m of L that intersects R_i , m intersects both a_i and b_i . Therefore, using the Klein model of \mathbb{H}^2 , one can easily see that R_i is a flow box. The height of the flow box is $\mu(a_i) = \mu(b_i)$. Since no leaf of L has a positive weight, we can assume that the height is less than $\epsilon/2$, again by choosing l_i sufficiently close to l . Hence, $R_1 \cup R_2 \cup s'$ is a flow box neighborhood of s with height less than ϵ . \square

Remark: Under the projection map from (\mathbb{H}^2, L) to its dual tree, the flow box neighborhood of height less than ϵ projects to a geodesic segment of length less than ϵ .

3.5. Approximating an Irrational Lamination. (For details, see [CEG, I.4.2.15].) Let S be a closed (orientable) hyperbolic surface, and

$L = (\lambda, \mu)$ be an irrational minimal measured lamination on S . We shall construct a sequence of simple geodesic loops that approximates $|\lambda|$.

Pick a leaf l of λ and a point P on l . Let s be a geodesic segment transversally intersecting l at P . Recall that $s \cap |\lambda|$ is a perfect subset of s . In addition, l is a dense subset of $|\lambda|$. Therefore, if we walk along l , starting from P , we never return to P , but we pass by P arbitrarily closely. More precisely, there exists a sequence of points P_i in $(l \cap s) \setminus \{P\}$, such that $\lim_{i \rightarrow \infty} \text{length}(a_i) = \infty$ and $\lim_{i \rightarrow \infty} \text{length}(b_i) = 0$, where a_i is a geodesic segment in l connecting P and P_i and b_i is the geodesic segment on S realizing the distance between P and P_i on S (Figure 2). Since $\lim_{i \rightarrow \infty} \text{length}(b_i) = 0$ and $\text{length}(b_i) \neq 0$ for all i , by taking a subsequence if necessary, we can assume that a_i and b_i intersect only at their end points, so that $a_i \cup b_i$ is a simple loop on S . Homotope this simple loop to a simple geodesic loop l_i on S . It turns out that (l_i) limits to $|\lambda|$ in the Chabauty topology. Besides, l_i transversally intersects $|\lambda|$ (uncountably many times).

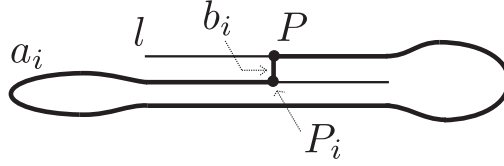


FIGURE 2.

Let m and n be geodesics on a complete hyperbolic surface. Assume that $m \cap n \neq \emptyset$, and pick $p \in m \cap n$. We can “rotate” m to n about p by a unique angle in $(-\pi/2, \pi/2]$. More precisely, we do the following: Let \tilde{p} be a lift of p to \mathbb{H}^2 , and let \tilde{m} and \tilde{n} be the lifts of m and n , respectively, to \mathbb{H}^2 that intersect at \tilde{p} . Then, we can indeed rotate \tilde{m} to \tilde{n} about \tilde{p} by a unique angle in $(-\pi/2, \pi/2]$. Let $\angle_p(m, n)$ denote this angle, and call it the *angle* between m and n at p .

Let ν be a geodesic lamination on the surface that intersects n .

Definition 3.2. *The angle between ν and n is*

$$\angle(\nu, n) = \sup\{ |\angle_p(m, n)| \mid m \in \nu \text{ and } p \in m \cap n \} \in [0, \pi/2].$$

Lemma 3.3. $\lim_{i \rightarrow \infty} \angle(\lambda, l_i) = 0$ and $\lim_{i \rightarrow \infty} \mu(l_i) = 0$.

Proof. Choose $x_i \in |\lambda| \cap l_i$, and let m_i be the leaf of λ that intersects l_i at x_i . Let $\theta_i = \angle_{x_i}(m_i, l_i)$. Note that $|\lambda|$ is a compact subset of S . Therefore, by taking a subsequence if necessary, (x_i, θ_i) converges to

$(x, \theta) \in |\lambda| \times [-\pi/2, \pi/2]$. Assume that $\lim_{i \rightarrow \infty} \angle(\lambda, l_i) \neq 0$. Then there exists a sequence (x_i, θ_i) converging to (x, θ) with $\theta \neq 0$. The sequence (l_i) converges to a geodesic that intersects $|\lambda|$ at x with angle θ . This contradicts the convergence of (l_i) to $|\lambda|$.

Next, we prove that $\lim_{i \rightarrow \infty} \mu(l_i) = 0$. Let $c_i = a_i \cup b_i$, which is a simple loop on S . Observe that $\lim_{i \rightarrow \infty} \mu(c_i) = 0$, since $\mu(a_i) \equiv 0$ and $\lim_{i \rightarrow \infty} \mu(b_i) = 0$. Therefore, it suffices to show that $\mu(l_i) \leq \mu(c_i)$. The basic idea is that a geodesic loop realizes the minimal transversal measure among all the loops in the same homotopy class. Let $\gamma_i \in \pi_1(S)$ be the homotopy class of l_i . Regarding l_i as a bi-infinite geodesic, let \tilde{l}_i be the lift of l_i , such that γ_i translates \mathbb{H}^2 along the geodesic \tilde{l}_i by $\text{length}(l_i)$. Regarding l_i as a simple closed path, let \bar{l}_i be a lift of l_i , such that \bar{l}_i is contained in \tilde{l}_i . Then γ_i identifies the ends of \bar{l}_i . Recall that \mathcal{P} is the projection from \mathbb{H}^2 to the \mathbb{R} -tree T dual to \tilde{L} (§3.3). Then $\mathcal{P}(l_i)$ is a geodesic in T , and the action of γ_i on T isometrically translates along $\mathcal{P}(l_i)$ by $\mu(l_i) = \text{length}(\mathcal{P}(\bar{l}_i))$. Similarly, regarding c_i as a simple closed path, let \bar{c}_i be a lift of c_i to \mathbb{H}^2 , such that the ends of \bar{c}_i are identified by γ_i . Then $\mathcal{P}(\bar{c}_i)$ is a piecewise geodesic path in T , and γ_i identifies the ends of $\mathcal{P}(\bar{c}_i)$. We also have $\mu(c_i) = \text{length}(\mathcal{P}(\bar{c}_i))$. The translation length of γ_i is equal to or less than the distance between the ends of $\mathcal{P}(\bar{c}_i)$ since the translation length of γ_i is $\inf\{\text{dist}_T(x, \gamma_i(x)) \mid x \in T\}$. Since γ_i translates along $\mathcal{P}(\bar{l}_i)$, the translation length is $\text{length}(\mathcal{P}(\bar{l}_i))$. Hence, $\mu(l_i) \leq \mu(c_i)$. \square

3.6. Bending Maps. (For details, see [EM].) Let $L = (\lambda, \mu)$ be a measured lamination on \mathbb{H}^2 . Then L induces a *bending map* $\beta_L = \beta : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ by “bending \mathbb{H}^2 inside \mathbb{H}^3 along λ by angle μ ”. The bending map β_L is continuous and unique up to the post-composition with an element of $\text{PSL}(2, \mathbb{C})$. In addition, β_L is isometric on each leaf of λ and on the closure of each complementary region of $|\lambda|$. The map β_L is C^1 -smooth in the complement of the leaves of L with positive weight (note that the complement has the full Lebesgue measure). Roughly speaking, if $x, y \in \mathbb{H}^2$ are sufficiently close to each other, then the hyperbolic tangent planes of β_L at x and y intersect at the external angle approximately equal to $\mu(\overline{xy})$ (with respect to the normal vector field of β ; c.f. the hyperbolic tangent planes defined in §3.8). Assume that l is an isolated leaf of λ and that Q and R are two adjacent complementary regions of $|\lambda|$ separated by l . Then $\beta_L(cl(Q))$ and $\beta_L(cl(R))$ are isometric copies of $cl(Q)$ and $cl(R)$ that intersect at the external angle $w(l)$. This property determines the bending map β_L if L consists of isolated leaves. For general L , there is a sequence

(L_i) of measured laminations on \mathbb{H}^2 consisting of finitely many leaves, that approximates L in Thurston's topology. Then $\beta_L = \lim_{i \rightarrow \infty} \beta_{L_i}$ uniformly on compacts.

3.7. Convex Hull Boundaries. (For details, see [EM].) Let X be a simply connected (open) region in $\hat{\mathbb{C}}$. Then X can be regarded as a projective structure on $\mathring{\mathbb{D}}^2$. Consider $\text{Conv}(\hat{\mathbb{C}} \setminus X)$, the convex hull of $\hat{\mathbb{C}} \setminus X$ in \mathbb{H}^3 . It turns out that $\partial \text{Conv}(\hat{\mathbb{C}} \setminus X)$ is isometric to \mathbb{H}^2 with respect to the induced path metric on $\partial \text{Conv}(\hat{\mathbb{C}} \setminus X)$. There is a unique measured lamination L on \mathbb{H}^2 such that L does not contain leaves of weight more than π and its bending map β_L realizes the isometry from \mathbb{H}^2 to $\partial \text{Conv}(\hat{\mathbb{C}} \setminus X)$. Then, by the orthogonal projection along geodesics in \mathbb{H}^3 , X maps onto $\text{Im}(\beta_L)$.

3.8. Thurston's Parameterization of Projective Structures. Let $P(S)$ be the space of all projective structures on S . William Thurston gave a parametrization of $P(S)$ that reflects the geometry of projective structures in a combinatorial manner. This parametrization is useful for the proof of the main theorems of this paper, since it involves a decomposition of \tilde{S} into f -injective regions, where $f : \tilde{S} \rightarrow \hat{\mathbb{C}}$ is the developing map of a projective structure.

Theorem 3.4 (Thurston). *$P(S)$ is naturally homeomorphic to the product of the Teichmüller Space of S and the space of measured laminations on S :*

$$(1) \quad P(S) \simeq \mathcal{T}(S) \times \mathcal{ML}(S) (\simeq \mathbb{R}^{6g-6} \times \mathbb{R}^{6g-6})$$

(The proof is in [KT]. For the following discussion, see also [Ta].) Below we discuss some properties of this homeomorphism. An element in the left hand side of (1) is a pair (f, ρ) , where $f : \tilde{S} \rightarrow \hat{\mathbb{C}}$ and $\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$. Take a pair $(\tau, L) \in \mathcal{T}(S) \times \mathcal{ML}(S)$. Then the total lift $(\mathbb{H}^2, \tilde{L})$ of (τ, L) induces a bending map $\beta : \mathbb{H}^2 \rightarrow \mathbb{H}^3$. Furthermore, since the action of $\pi_1(S)$ preserves \tilde{L} , β induces a representation $\rho_L : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ such that β is ρ_L -equivariant. If (f, ρ) and (τ, L) represent the same projective structure, then $\rho = \rho_L$. Letting $\rho_{id} : \pi_1(\mathring{\mathbb{D}}^2) \rightarrow \text{PSL}(2, \mathbb{C})$ be the trivial representation, (f, ρ_{id}) is a projective structure on $\mathring{\mathbb{D}}^2$, which is the universal cover of C . Moreover, (f, ρ_{id}) corresponds to the measured lamination $(\mathbb{H}^2, \tilde{L})$ through the orthogonal projection and the bending map, which generalizes the correspondence between a simply connected region in $\hat{\mathbb{C}}$ and an injective bending map discussed in §3.7. Namely, in our current case, f and β are not necessarily embeddings, and we need to divide the domain

of f and the domain of β so that their corresponding subdomains are homeomorphic through the orthogonal projection.

We shall first discuss the same correspondence for projective structures on $\mathring{\mathbb{D}}^2$, which is more general than the above case. Namely, there is a bijective correspondence between the projective structures on $\mathring{\mathbb{D}}^2$ (that are not conformally equivalent to the Euclidian plane) and the measured laminations on \mathbb{H}^2 (up to the action of $\mathrm{PSL}(2, \mathbb{R})$). For a measured lamination $L = (\lambda, \mu)$ on \mathbb{H}^2 , let $C = C(L) = (f, \rho_{id})$ denote the corresponding projective structure on $\mathring{\mathbb{D}}^2$. We shall discuss the correspondence between $C(L)$ and L . There are a (topological) measured lamination $L' = (\lambda', \mu')$ on $(\mathring{\mathbb{D}}^2, C)$ and the *collapsing map* $\kappa : (\mathring{\mathbb{D}}^2, C, L') \rightarrow (\mathbb{H}^2, L)$, which describe the subdivision and the orthogonal projections. For each leaf l of L with positive weight, $\kappa^{-1}(l)$ is a crescent of angle $w(l)$ with the canonical foliation (compare [Ka2, 11.12]). Conversely, each crescent of angle h in (S, C) projects to a leaf of weight h via κ in the way discussed in §3.2. In the complement of such crescents, κ is an isomorphism, i.e. a C^1 -diffeomorphism that preserves the measured lamination. In summary, L' is topologically obtained from L by blowing up each leaf l with positive weight of L as above. (Note that there is no periodic leaves with positive weight of L' .) The collapsing map κ is a continuous surjective map that homeomorphically takes each leaf of L' to a leaf of L and each component of $(\mathring{\mathbb{D}}^2, C) \setminus |L'|$ to a component of $(\mathbb{H}^2, L) \setminus |L|$. Furthermore, this correspondence is bijective except the correspondence between the leaves of the crescents and the leaves of positive weight.

A *maximal ball* of a projective structure C on $\mathring{\mathbb{D}}^2$ is a maximal open subset of $\mathring{\mathbb{D}}^2$ that f homeomorphically takes to a round open disc in $\hat{\mathbb{C}}$, where the maximality is defined with respect to the set inclusion. If U is a maximal ball, then $\partial f(U)$ is a round circle in $\hat{\mathbb{C}}$, and $\mathrm{Conv}(\partial f(U)) \subset \mathbb{H}^3$ is a copy of \mathbb{H}^2 whose ideal boundary is $\partial f(U)$. Let $H_U = \mathrm{Conv}(\partial f(U))$, and let $\Psi_U : f(U) \rightarrow H_U$ be the orthogonal projection along geodesics in \mathbb{H}^3 .

Let R be the closure of a component of $\mathring{\mathbb{D}}^2 \setminus |L'|$, or a leaf of L' that does not bound a component of $\mathring{\mathbb{D}}^2 \setminus |L'|$. Then, R is contained in a unique maximal ball U and $R = R_U$ is called the *core* of U . Conversely, each maximal ball U contains a unique core. These cores of maximal balls form a partition of $\mathring{\mathbb{D}}^2$. Let $\beta : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by L . Then we have $\Psi_U \circ f = \beta \circ \kappa$ on each core R_U , which describes the correspondence of f and β . Define $\Psi : (\mathring{\mathbb{D}}^2, C) \rightarrow \mathbb{H}^3$ by $\Psi(x) = \Psi_U(x)$ when $x \in R_U$. Then we have $\Psi = \beta \circ \kappa$.

Let W be the union of leaves of L with positive weight. Recall that β is C^1 -smooth except on W .

Definition 3.5. *The hyperbolic tangent plane of Ψ at x is $H_U = \partial \text{Conv}(f(U)) \cong \mathbb{H}^2$ when $x \in R_U$ (see Figure 3).*

This tangent plane is a support plane of $\beta(U_{\kappa(x)})$ at $\Psi(x)$ where $U_{\kappa(x)}$ is a sufficiently small neighborhood of $\kappa(x)$. Then this hyperbolic tangent plane coincides with the standard hyperbolic tangent plane at each point of $\tilde{S} \setminus \tilde{\kappa}^{-1}(W)$, which is the complement of the disjoint foliated crescents. When $x \in \tilde{S}$ moves infinitesimally, the hyperbolic tangent plane of Ψ at x rotates about $\beta(l_x)$ in \mathbb{H}^2 by the amount of the transversal measure μ' , where l_x is a leaf of L' through x if it exists. In particular, when x moves along a leaf or a moves in the closure of a component of $\tilde{S} \setminus |L'|$, then the hyperbolic tangent plane does not change.

Below we show that the hyperbolic tangent planes depend continuously on $x \in \tilde{S}$. Since β is C^1 -smooth away from the isolated leaves of \tilde{L} , continuity of the hyperbolic tangent planes on the complement to the union of crescents is clear. Consider a crescent $C_l = \kappa^{-1}(l)$, where l is an isolated leaf of \tilde{L} . Let H_i ($i = 1, 2$) denote the closures of the components of $\mathbb{H}^2 \setminus l$. For each $i = 1, 2$, the restriction $\beta|_{H_i}$ is C^1 -smooth along l . Therefore, there is a unique hyperbolic plane in \mathbb{H}^3 through $\beta(x)$ which shares the tangent plane with the surface $\beta(H_i)$ at the point $\beta(x)$. These hyperbolic tangent planes of $\beta(H_i)$ at $\beta(x)$ intersect along the geodesic $\beta(l)$ at angle $w(l)$. On the other hand, these hyperbolic tangent planes of $\beta(H_i)$ agree with the hyperbolic tangent planes of Ψ at points on their corresponding boundary leaves of C_l . For each $x \in C_l$, the hyperbolic tangent plane of Ψ at x contains $\beta(l)$. When x moves in C_l transversally to the foliation, the hyperbolic tangent plane rotates about $\beta(l)$ by the amount of the transversal measure μ' . Therefore, the hyperbolic tangent plane of Ψ depends continuously on $x \in C_l$, hence depends continuously on $x \in \tilde{S}$.

Let us return to the correspondence between a measured lamination L on (S, τ) and a projective structure $C = (f, \rho)$ on S . Since $\tilde{S} \cong \mathbb{D}^2$, we have the canonical lamination \tilde{L}' on (\tilde{S}, \tilde{C}) and the collapsing map $\tilde{\kappa} : (\tilde{S}, \tilde{C}, \tilde{L}') \rightarrow (\mathbb{H}^2, \tilde{L})$. By its construction, \tilde{L}' is invariant under the action of $\pi_1(S)$, and it induces a measured lamination $L' = \tilde{L}'/\pi_1(S)$ on S . In addition, $\tilde{\kappa}$ is ρ -equivariant, and it induces the *collapsing map* $\kappa : (S, C, L') \rightarrow (S, \tau, L)$. Accordingly, for each periodic leaf l of L , $\kappa^{-1}(l)$ is a flat cylinder of height $w(l)$ with the canonical foliation. Conversely, each foliated flat cylinder of height h in (S, C) projects

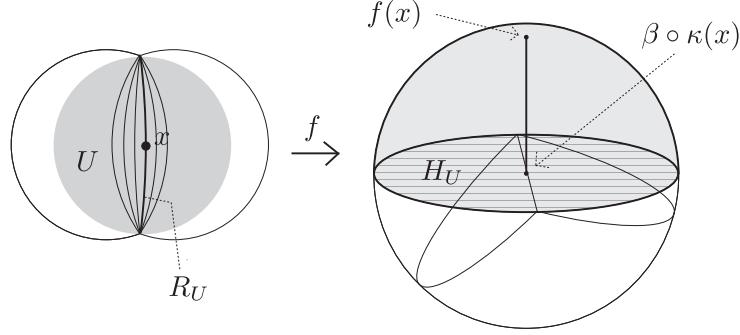


FIGURE 3.

to a periodic leaf of weight h via κ . In the complement of such flat cylinders, κ is an isomorphism.

Figure 4 illustrates the basic case when we have a measured lamination consisting of a periodic leaf on a complete hyperbolic cylinder. Near a periodic leaf of L on (S, τ) , we locally have a similar diagram.

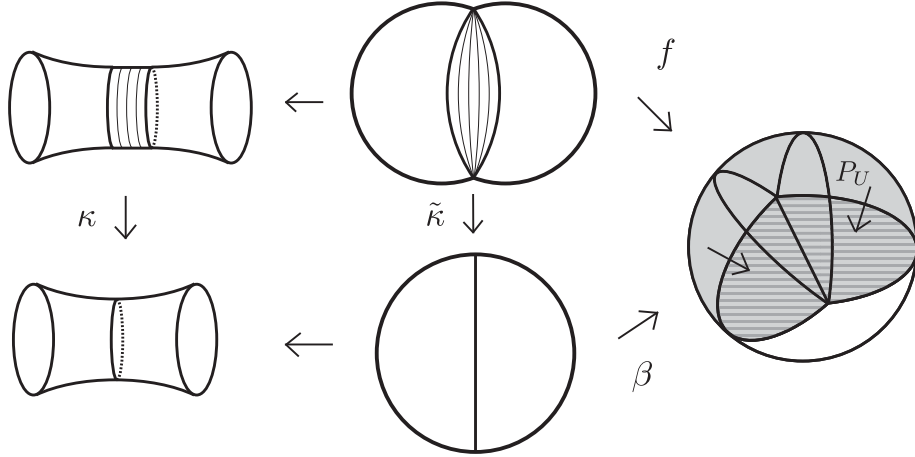


FIGURE 4.

3.8.1. *The Intersection of a Lamination and a Convex Set in \mathbb{H}^2 .* Let $L = (\lambda, \mu)$ be a measured lamination on \mathbb{H}^2 .

Definition 3.6. Let X be a geodesic or a convex subset of \mathbb{H}^2 bounded by geodesics. The **intersection** of L and X is a measured lamination (λ_X, μ_X) on \mathbb{H}^2 , where $\lambda_X = \text{cl}\{l \in \lambda \mid l \cap X \neq \emptyset\}$ and μ_X is the restriction of μ to λ_X . Denote the intersection by $I(L, X)$.

Definition 3.7. Let X be a convex subset of \mathbb{H}^2 bounded by geodesics. The **restriction** of L to X is a measured lamination $(\lambda|_X, \mu|_X)$ on X , where $\lambda_X = \{l \cap X \mid l \in \lambda\}$ and $\mu|_X$ is defined by $\mu|_X(s) = \mu(s)$ for all geodesic segments s in X . Denote the restriction by $L|_X$.

Definition 3.8. Let C_1 be a projective structure on a surface F_1 , and let F_2 be a subsurface of F_1 . Then the **restriction** of C_1 to F_2 is the restriction of the atlas of C_1 to F_2 . Let $\mathcal{R}(C_1, F_2)$ denote the restriction of C_1 to F_2 . Conversely, if a projective structure C_3 on a surface F_3 is isomorphic to the one obtained by restricting C_1 to a subsurface of F_1 , then we say that C_3 **embeds** into C_1 .

Let $C_1 = (f_1, \rho_1)$ and assume, in addition, that the inclusion $F_2 \subset F_1$ is π_1 -injective. Then the above definition is equivalent to the following: The restriction $\mathcal{R}(C_1, F_2)$ is the projective structure $(f_1|_{\tilde{F}_2}, \rho_1|_{\pi_1(F_2)})$, where $f_1|_{\tilde{F}_2}$ is the restriction of f_1 to a lift \tilde{F}_2 of F_2 to the universal cover of F_1 and $\rho_1|_{\pi_1(F_2)}$ is the restriction of ρ_1 to $\pi_1(F_2)$ acting on \tilde{F}_2 .

Let L be a measured lamination on \mathbb{H}^2 , and let $C(L) = (f_L, \rho_{id})$, the projective structure on $\mathring{\mathbb{D}}^2$ corresponding to L (§3.8). Let X be a convex subset of \mathbb{H}^2 bounded by geodesics, allowing X to possibly be a geodesic. Let $I = I(L, X)$, and let $C(I) = (f_I, \rho_{id})$. We also let κ_L and $\kappa_I : \mathring{\mathbb{D}}^2 \rightarrow \mathbb{H}^2$ be the collapsing maps for $C(L)$ and $C(I)$, respectively.

Lemma 3.9. *There exists a homeomorphism $\phi : \kappa_I^{-1}(X) \rightarrow \kappa_L^{-1}(X)$ such that $f_I = f_L \circ \phi$ on $\kappa_I^{-1}(X)$. Moreover, $C(I)$ embeds into $C(L)$.*

Proof. Consider the leaves of L and components of $\mathbb{H}^2 \setminus |L|$ that intersect X , and let X' be the union of these leaves and components. Then X' is a convex subset of \mathbb{H}^2 containing X , and it is bounded by some leaves of L . We also have $I = I(L, X) = I(L, X')$. Therefore, it suffices to prove the lemma for X' . Let L' and I' be the canonical measured laminations on $(\mathring{\mathbb{D}}^2, C(L))$ and $(\mathring{\mathbb{D}}^2, C(I))$, respectively.

Since $L|_{X'} = I|_{X'}$, we can assume that $\beta_L = \beta_I$ on X' , where β_L and β_I are the bending maps induced by L and I , respectively. Therefore, $\beta_L = \beta_I$ on $cl(X')$ by the continuity of bending maps. Recall that $\kappa_I^{-1}(X')$ and $\kappa_L^{-1}(X')$ are obtained from X' in the exactly same way, namely by blowing up the periodic leaves of $L|_{X'} = I|_{X'}$. Therefore, we have a canonical homeomorphism $\phi : cl(\kappa_I^{-1}(X')) \rightarrow cl(\kappa_L^{-1}(X'))$ such that ϕ isomorphically takes $I'|_{\kappa_I^{-1}(X')}$ to $L'|_{\kappa_L^{-1}(X')}$ and $\kappa_I = \kappa_L \circ \phi$ on $cl(\kappa_I^{-1}(X'))$. Furthermore, the hyperbolic tangent plane of $\beta_I \circ \kappa_I$ at $x \in cl(\kappa_I^{-1}(X'))$ coincides with the hyperbolic tangent plane of $\beta_L \circ \kappa_L$ at $\phi(x)$. (See §3.8.) The maximal ball of C_I whose core contains x maps to a round open disk by f_L , and the maximal ball of C_L whose

core contains $\phi(x)$ maps to a round open disk by f_I . The convex hull boundaries of these open disks are the hyperbolic tangent planes of $\beta_L \circ \kappa_L$ at $\phi(x)$ and of $\beta_I \circ \kappa_I$ at x , and therefore, they must agree. Then the round disks on $\hat{\mathbb{C}}$ also coincide. Recall that $f_I(x)$ and $\beta_I \circ \kappa_I(x)$ are connected by a geodesic in \mathbb{H}^3 orthogonal to the hyperbolic tangent plane, and so are $f_L(\phi(x))$ and $\beta_L \circ \kappa_L(\phi(x))$. Hence, $f_I(x) = f_L(\phi(x))$.

Each component H of $\mathbb{H}^2 \setminus X'$ is an open or closed half plane bounded by a leaf l of L , and it does not contain leaves of I . Then $\kappa_L^{-1}(H)$ is a component of $\mathring{\mathbb{D}}^2 \setminus \kappa_L^{-1}(X')$. This component is simply connected and bounded by a leaf l_L of L' that maps to l via κ_L . Similarly, $\kappa_I^{-1}(H)$ is a component of $\mathring{\mathbb{D}}^2 \setminus \kappa_I^{-1}(X')$. This component is simply connected and bounded by a boundary curve l_I of $\kappa_I^{-1}(X')$. In addition, l_I is a leaf of I' or contained in a component of $\mathring{\mathbb{D}}^2 \setminus |I'|$, and it is homeomorphic to l_L and l via ϕ and κ_I , respectively. The leaf l_L is contained in a unique maximal ball U of $C(L)$, whose convex hull boundary is the hyperbolic tangent plane of $\beta_L \circ \kappa_L$ at each point in l_L . Similarly, l_I is contained in a unique maximal ball V of $C(I)$, whose convex hull boundary is the hyperbolic tangent plane of $\beta_I \circ \kappa_I$ at each point in l_I . Thus these hyperbolic tangent planes are the same planes in \mathbb{H}^3 . Therefore, we can identify V and U by a C^1 -diffeomorphism $\psi : V \rightarrow U$ such that $f_I = f_L \circ \psi$ on V . Then $\psi = \phi$ on $\kappa_I^{-1}(X') \cap V$. Therefore, the embedding $\phi : \kappa_I^{-1}(X) \rightarrow \kappa_L^{-1}(X) \subset \mathring{\mathbb{D}}^2$ extends to $\kappa_I^{-1}(H)$, preserving $f_I = f_L \circ \phi$. The different components of $\mathring{\mathbb{D}}^2 \setminus \kappa_I^{-1}(X')$ map to different components of $\mathring{\mathbb{D}}^2 \setminus \kappa_L^{-1}(X')$ by the extension. Hence, since $\kappa_I^{-1}(\mathbb{H}^2) = \mathring{\mathbb{D}}^2$, we have an embedding of $C(I)$ into $C(L)$. \square

Assume that X has non-empty interior and no boundary geodesic of X transversally intersects a leaf of L with positive weight. Then $cl(\kappa_I^{-1}(X))$ and $cl(\kappa_L^{-1}(X))$ are C^1 -smooth subsurfaces of $\mathring{\mathbb{D}}^2$. Thus we immediately obtain

Corollary 3.10. *$\mathcal{R}(C(I), cl(\kappa_I^{-1}(X)))$ and $\mathcal{R}(C(L), cl(\kappa_L^{-1}(X)))$ are isomorphic as projective structures.*

4. THE INTERSECTION OF A LAMINATION AND ITS APPROXIMATING LOOP

Let $L = (\lambda, \mu)$ be an irrational minimal lamination on a closed orientable hyperbolic surface S , and (l_i) be the sequence of simple loops on S that converges to $|\lambda|$ constructed in §3.5. Let $\tilde{L} = (\tilde{\lambda}, \tilde{\mu})$ be the total lift of L to \mathbb{H}^2 , and let \tilde{l}_i be a lift of l_i to \mathbb{H}^2 . Let $L_i = (\lambda_i, \mu_i)$ be $I(\tilde{L}, l_i)$ (see §3.8.1). Note that the dual tree of L_i is isometric to \mathbb{R} .

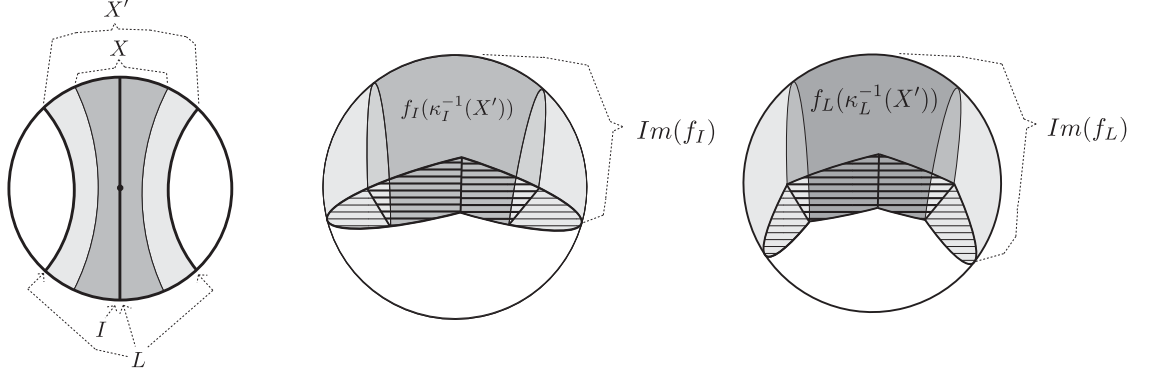


FIGURE 5. A basic example for Lemma 3.9.

Definition 4.1. Let $M = (\nu, \omega)$ be a measured lamination on a hyperbolic surface F . Define the **norm** of M by

$$\|M\| = \sup\{\omega(s)\},$$

where s varies over all geodesic segments of length less than 1 on F .

We next prove that the transversal measure μ_i of short geodesic segments is bounded by an arbitrary small number, provided that i is large:

Proposition 4.2. $\lim_{i \rightarrow \infty} \|L_i\| = 0$.

The basic idea of the proof is that, when a geodesic segment s with $\text{length}(s) < 1$ intersects a measured lamination at an angle close to zero, its transversal measure is also close to zero.

Let x be a point on a leaf l of $\tilde{\lambda}$. For $\theta \in (-\pi/2, \pi/2]$, let $l_{x,\theta}$ be the geodesic on \tilde{S} intersecting l at x with $\angle_x(l, l_{x,\theta}) = \theta$ (see §3.5). Set $I(\tilde{L}, l_{x,\theta}) = (\lambda_{x,\theta}, \mu_{x,\theta})$.

Lemma 4.3. For every $\epsilon > 0$, there exists $\theta_0 > 0$ such that, if $\theta \in (-\theta_0, \theta_0)$ and $x \in |\tilde{\lambda}|$, then $\mu_{x,\theta}(s) < \epsilon$ for all geodesic segments s in \mathbb{H}^2 with $x \in s$ and $\text{length}(s) < 1$.

Proof. For an arbitrary y in $|\tilde{\lambda}|$, let l be the leaf of $\tilde{\lambda}$ through y . Then consider $D_2(y)$, the closed hyperbolic disk of radius 2 centered at y . In each component of $\mathbb{H}^2 \setminus l$, choose a geodesic g_i ($i = 1, 2$) close to l that does *not* transversally intersect a leaf of $\tilde{\lambda}$, i.e. g_i is a leaf of $\tilde{\lambda}$ or is in the complement of $|\tilde{\lambda}|$. Let $R \subset \mathbb{H}^2$ be the convex region bounded by g_1 and g_2 , which contains l . For every $\epsilon > 0$, by applying Lemma 3.1 to $l \cap D_2(y)$, we can assume that g_1 and g_2 are close enough to l , so that the $R \cap D_2(y)$ is contained in a flow box of height less than ϵ .

Take a neighborhood U of y whose closure is contained in the interior of $R \cap D_1(y)$. Then there exists (small) $\theta_0 > 0$ such that, if $x \in U$ and $\theta \in (-\theta_0, \theta_0)$, then $l_{x,\theta} \subset R$ (see Figure 6). Since $R \cap D_2(y)$ is contained in the flow box of height less than ϵ and R supports $I(\tilde{L}, R)$, for every geodesic segment s in $D_2(y)$, the transversal measure of s with respect to $I(\tilde{L}, R)$ is bounded by ϵ . Therefore, since $I(\tilde{L}, l_{x,\theta})|_{D_2(y)}$ is a sublamination of $I(\tilde{L}, R)|_{D_2(y)}$, $\mu_{x,\theta}(s) < \epsilon$. If s is a geodesic segment in \mathbb{H}^2 such that $s \cap U \neq \emptyset$ and $length(s) < 1$, then $s \subset D_2(y)$. Thus, $\mu_{x,\theta}(s) < \epsilon$. This proves the lemma if x is in U , which is a neighborhood of y . Since S is compact and \tilde{L} is invariant under the deck transformations, the lemma follows. \square

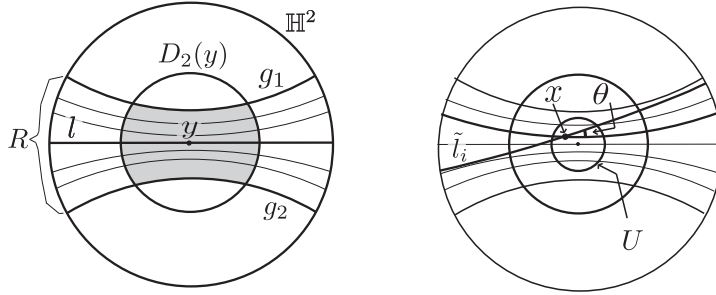


FIGURE 6. In the left picture, $R \cap D_2(y)$ is shaded.

Proof. (Proposition 4.2) Fix arbitrary $\epsilon > 0$. It suffices to show that, for sufficiently large i , if a geodesic segment s in \mathbb{H}^2 satisfies $\mu_i(s) \geq \epsilon$, then $length(s) \geq 1$. By Lemma 4.3, there exists $\theta_0 > 0$, such that, if $x \in |\tilde{\lambda}|$ and $\theta \in (-\theta_0, \theta_0)$, then $\mu_{x,\theta}(s) < \frac{2}{5}\epsilon$ for every geodesic segment s with $x \in s$ and $length(s) < 1$. By Lemma 3.3, for sufficiently large i , $\angle(\tilde{\lambda}, \tilde{l}_i) = \angle(\lambda, l_i) < \theta_0$ and $\mu(l_i) < \epsilon/2$.

Consider the cyclic subgroup of $\text{PSL}(2, \mathbb{R})$ generated by the translation along \tilde{l}_i by $length(l_i)$, which we can regard as $\pi_1(l_i) \subset \pi_1(S)$ acting on \mathbb{H}^2 . Choose a leaf $m \in \lambda_i$, and consider the orbit of m under the action of the cyclic group. This orbit forms a sublamination ν of λ_i . The leaves of ν intersect \tilde{l}_i at a constant angle less than θ_0 , and the distance between their consecutive intersection points equals to $length(l_i)$. Since $\mu(l_i) < \epsilon/2$ and the dual tree of L_i is isometric to \mathbb{R} , if a geodesic segment s lies between two adjacent leaves of ν , then $\mu_i(s) < \epsilon/2$. Therefore, if $\mu_i(s) \geq \epsilon$, then s transversally intersects $|\nu|$ at least three times. Let a_1, a_2, \dots, a_p be the intersection points lying on s in this order, and let A_1, A_2, \dots, A_p be the leaves of ν through these points.

Take $q \in \mathbb{N}$ such that $2q + 1$ is the maximal odd integer *not* exceeding p . Let r be the subsegment of s with end points a_1 and a_{2q+1} . (See Figure 7.) Then $\text{length}(r) \leq \text{length}(s)$ and $\frac{2}{5}\mu_i(s) \leq \mu_i(r) \leq \mu_i(s)$. Let r' be the geodesic segment that realizes the distance between A_1 and A_{2q+1} . Then r' is orthogonal to A_1 and A_{2q+1} . In addition, r' intersects \tilde{l}_i transversally, since otherwise $\tilde{l}_i, r', A_1, A_{2q+1}$ bound a hyperbolic triangle whose interior angle sum is π or a hyperbolic rectangle whose interior angle sum is 4π . Therefore, the triangle bounded by \tilde{l}_i, r', A_1 is isometric to the triangle bounded by $\tilde{l}_i, r', A_{2q+1}$. Thus $A_{q+1} \cap \tilde{l}_i$ is the middle point of r' . Note that $A_{q+1} \cap \tilde{l}_i \in r'$, \tilde{l}_i intersects $|\nu| \subset |\tilde{\lambda}|$ at $A_{q+1} \cap \tilde{l}_i$ at an angle less than θ_0 , and $\mu_i(\gamma') = \mu_i(\gamma) \geq \frac{2}{5}\epsilon$. Therefore, $\text{length}(r') \geq 1$. Hence, $1 \leq \text{length}(r') \leq \text{length}(r) \leq \text{length}(s)$. \square

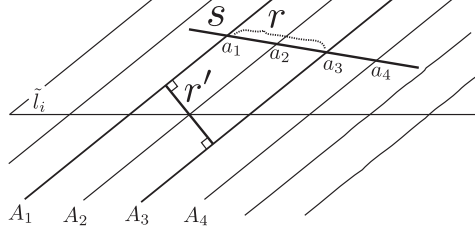


FIGURE 7. A case that $p = 4$ and $q = 1$.

5. INJECTIVITY OF BENDING MAPS

In this section, let $L = (\lambda, \mu)$ be a measured lamination on \mathbb{H}^2 and $\beta_L = \beta : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by L . Recall that $\|L\| = \sup\{\mu(s)\}$, where s varies over all geodesic segments on \mathbb{H}^2 of length less than 1.

Theorem 5.1 (Epstein, Marden and Markovic; [EMM], Theorem 4.2.2). *There exists $\delta \in (0, \pi)$ such that, if $\|L\| < \delta$, then the induced bending map β_L is a bilipschitz embedding; hence, it continuously extends to $\partial\mathbb{H}^2$ as an embedding whose image is a simple loop in $\partial\mathbb{H}^3$.*

To prove the decomposition theorem (Theorem 7.1), we need a generalization of Theorem 5.1. However, if we do not require flat cylinders in Theorem 7.1 to be integral, then Theorem 5.1 is sufficient. To state the generalization, let us set up some notions. A leaf l of L is *outermost*, if the other leaves of L lie only in one component of $\mathbb{H}^2 \setminus l$. Consider all outermost leaves of L with positive weight. Then this set forms a sublamination $\partial\lambda$ of λ , and $\partial\lambda$ consists of isolated leaves. Let ∂L be the measured lamination on \mathbb{H}^2 obtained by assigning each leaf of $\partial\lambda$ its

positive weight with respect to μ . Let $L' = (\lambda', \mu')$ be a sublamination of ∂L . We also let $d(L') = \inf\{\text{dist}_{\mathbb{H}^2}(l_1, l_2) \mid l_1, l_2 \in \lambda', l_1 \neq l_2\} \geq 0$. Let $L \setminus L' = (cl(\lambda \setminus \lambda'), \mu - \mu')$. Note that $|L'|$ bounds a convex region of \mathbb{H}^2 and that the convex region contains $|L \setminus L'|$.

Theorem 5.2. *For every $D > 0$, there exist $S, T > 0$ and $\delta \in (0, \pi)$ with the following property: If a measured lamination L on \mathbb{H}^2 contains a sublamination L' such that*

(i) $L' \subset \partial L$,

(ii) $\|L \setminus L'\| < \delta$,

(iii) $d(L') > D$, and

(iv) every leaf of L' has weight less than $\pi/2$,

then the induced bending map β_L is an injective (S, T) -quasiisometric embedding.

Proof. (Compare with the proof of [EMM, Theorem 4.2.2].) For arbitrary $D > 0$, choose $\theta_0 \in (0, \pi/2)$. Next, choose $\delta > 0$ small enough so that $\theta_0 + \delta < \pi/2$, $\sin(\theta_0) > \delta$ and $D(\delta - \sin(\theta_0)) + \delta < 0$. Then $\delta < \sin(\theta_0) < \sin(\pi/2 - \delta) = \cos(\delta)$. Let $p : [0, P] \rightarrow \mathbb{H}^2$ be an arbitrary geodesic segment parametrized by arc length, where P is the length of p . Let $\gamma = \beta_L \circ p : [0, P] \rightarrow \mathbb{H}^3$.

Discrete Case. First, assume that L consists of isolated leaves. Then p transversally intersects $|L|$ only finitely many times. Let $t_1 < t_2 < \dots < t_{n-1}$ be the points in $(0, P)$ that correspond to such transversal intersection points. The curve γ is a piecewise geodesic, and t_i are its singular points. In addition, let $t_0 = 0$ and $t_n = P$. For $i \in \{1, 2, \dots, n-1\}$, let l_i denote the leaf of L intersecting p at $p(t_i)$. For every pair of consecutive geodesic segments $\gamma([t_{i-1}, t_i])$ and $\gamma([t_i, t_{i+1}])$, their exterior angle, $\pi - \angle\gamma(t_{i-1})\gamma(t_i)\gamma(t_{i+1})$, is bounded from above by $\omega(l_i)$, by the triangle inequality.

Since $|L'|$ bounds a convex region that contains $|L \setminus L'|$, p transversally intersects $|L'|$ at most twice at t_1 and/or t_{n-1} . Define $d_p = d : [0, t_n] \rightarrow \mathbb{R}_{\geq 0}$ by $d(t) = \text{dist}_{\mathbb{H}^3}(\gamma(0), \gamma(t))$. Define $\theta_p = \theta : [0, t_n] \rightarrow [0, \pi]$ by $\theta(0) = 0$ and $\theta(t) = \angle\gamma(0)\gamma(t)\gamma(t_i)$ when t satisfies $t_i < t \leq t_{i+1}$ (Figure 8). Call θ_p the *angle function* for γ . Then $\theta(t)$ is smooth in (t_i, t_{i+1}) . At t_i with $0 < i < n$, θ is smooth from the left and discontinuous from the right. Let $\theta^+(t_i) = \lim_{t \searrow t_i} \theta(t)$. Then, by the bound for the exterior angle and the triangle inequality,

$$(2) \quad |\theta^+(t_i) - \theta(t_i)| \leq \omega(l_i).$$

We also have

$$d'(t) = \cos(\theta(t)), \text{ and } \theta'(t) = -\frac{\sin(\theta(t))}{\tanh(d(t))} \leq -\sin(\theta(t)) \leq 0$$

(see [CEG, I.4.2.12.], [EMM, 4.4.]). In particular, $\theta(t)$ does not increase on $(t_i, t_{i+1}]$. Note that θ' and $\theta^+(t_i) - \theta(t_i)$ completely determine the change in θ . Therefore, by (2), the total increase of θ from its discontinuity points in $[t, t+1] \cap [0, P]$ is at most $\|L\|$ for all t .

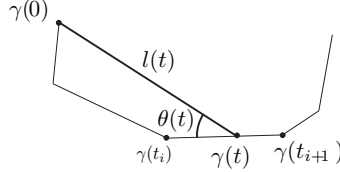


FIGURE 8.

Let $I_p = \{t \in [0, P] \mid \theta(t) > \theta_0 + \delta\}$, and let $\text{mes}(I_p)$ denote the Lebesgue measure of I_p . The following proposition yields good upper bounds for $\text{mes}(I_p)$ and θ_p .

Proposition 5.3. *There exist $B > 0$ and $0 < C < \pi s$, which depend only on D, θ_0 and δ , such that $\text{mes}(I_p) < B$ and $\theta_p < C$ for all geodesic segments $p : [0, P] \rightarrow \mathbb{H}^2$.*

Remark: B and C do not depend on L and L' as long as they satisfy the Hypotheses (i) - (iv) of Theorem 5.2.

The following two lemmas are used in the proof of the proposition. Let $[a, b)$ be a subinterval of $[0, P]$ in which p does *not* transversally intersect leaves of L' . Then $\|L \setminus L'\| < \delta$ bounds the total increase of θ from the discontinuity points in $[t, t+1] \cap [a, b]$ for all $t \in [a, b]$.

Lemma 5.4. *If $\theta(a) \leq \theta_0$, then $\theta(t) < \theta_0 + \delta$ for all $t \in [a, b]$.*

Proof. (This proof is essentially the same as the proof of Lemma 4.4 in [EMM].) Assume that θ is *not* bounded by $\theta_0 + \delta$. Then let $s_2 = \inf\{t \in [a, b] \mid \theta(t) \geq \theta_0 + \delta\}$, and let $s_1 = \max\{t \in [a, s_2] \mid \theta(t) < \theta_0\}$. Note that s_1, s_2 are both discontinuity points. Thus $\theta^+(s_2) - \theta(s_1) > (\theta_0 + \delta) - \theta_0 = \delta$. Then $\theta(s_2 + \epsilon) - \theta(s_1) > \delta$ for all sufficiently small $\epsilon > 0$. Therefore, $s_2 - s_1 \geq 1$. If $t \in (s_1, s_2)$, then $\theta_0 \leq \theta(t) < \theta_0 + \delta < \pi/2$. If, in addition, t is a smooth point, $\theta'(t) \leq -\sin(\theta(t)) \leq -\sin(\theta_0)$. Therefore,

$$\begin{aligned} \theta(s_1 + 1) &\leq \theta(s_1) + \|L \setminus L'\| + \int_{s_1}^{s_1+1} \theta'(t) \\ &< \theta(s_1) + \delta - \sin(\theta_0) < \theta(s_1) < \theta_0. \end{aligned}$$

This inequality contradicts the definition of s_1 . □

Lemma 5.5. *If $\theta(a) \leq \pi - (\theta_0 + \delta)$, then*

$$(3) \quad \begin{aligned} \theta(t) &< \max\{\theta_0 + \delta, \theta(a) - (t - a)(\sin(\theta_0) - \delta) + \delta\} \\ &< \pi - \theta_0, \end{aligned}$$

for all $t \in [a, b]$.

Remark: The right hand side of (3) decreases as t increases until it reaches $\theta_0 + \delta$.

Proof. As with Lemma 5.4, we can show that $\theta(t) < \theta(a) + \delta \leq \pi - \theta_0$ for all $t \in [a, b]$. If $\theta(t) \leq \theta_0$ for some $t \in [a, b]$, then let $s = \min\{t \in [a, b] \mid \theta(t) \leq \theta_0\}$. By Lemma 5.4, $\theta(t) < \theta_0 + \delta < \pi - \theta_0$ for all $t \in [s, b]$. For all $t \in [a, s)$, $\theta'(t) < -\sin(\theta_0)$ since $\theta_0 < \theta(t) < \pi - \theta_0$. Therefore,

$$\begin{aligned} (4) \quad \theta(t) &< \theta(a) + \int_a^t \theta'(u) du + (t - a + 1)\|L \setminus L'\| \\ (5) \quad &< \theta(a) - (t - a)\sin(\theta_0) + (t - a + 1)\delta \\ (6) \quad &< \theta(a) - (t - a)(\sin(\theta_0) - \delta) + \delta. \end{aligned}$$

If $\theta(t) > \theta_0$ for all $t \in [a, b]$, similarly (4) – (6) hold. This completes the proof. \square

Proof. (Proposition 5.3) Recall that the geodesic segment p transversally intersects $|L'|$ at most twice at t_1 and/or t_{n-1} . Thus we have four cases:

- Case 1.* p does not transversally intersect $|L'|$.
- Case 2.* p transversally intersects $|L'|$ only at t_1 .
- Case 3.* p transversally intersects $|L'|$ only at t_{n-1} .
- Case 4.* p transversally intersects $|L'|$ at both t_1 and t_{n-1} .

In every case, $\theta \equiv 0$ in $[0, t_1]$, and $I_p \cap [0, t_1] = \emptyset$.

Case 1. Since $\theta(0) = 0 < \theta_0$ and p does not transversally intersects $|L'|$, by applying Lemma 5.4 with $a = 0$ and $b = P$, we have $\theta(t) < \theta_0 + \delta < \pi/2$ for all $t \in [0, P]$. Therefore, $I_p = \emptyset$. Hence, we can take arbitrary $B \in \mathbb{R}_{>0}$ and $C = \theta_0 + \delta$.

Case 2. Since $w(l_1) < \pi/2$, $\theta^+(t_1) < \pi/2 < \pi - (\theta_0 + \delta)$. In $(t_1, t_n]$, by taking the limit of (3) as $a \searrow t_1$, we obtain

$$\begin{aligned} (7) \quad \theta(t) &\leq \max\{\theta_0 + \delta, \theta^+(t_1) - (t - t_1)(\sin(\theta_0) - \delta) + \delta\} \\ (8) \quad &< \pi - \theta_0. \end{aligned}$$

Then, if $\theta(t) > \theta_0 + \delta$,

$$\begin{aligned} (9) \quad \theta^+(t_1) - (t - t_1)(\sin(\theta_0) - \delta) + \delta &> \theta_0 + \delta, \\ (10) \quad \text{and} \quad t - t_1 &< \frac{\pi/2 - \theta_0}{\sin(\theta_0) - \delta}. \end{aligned}$$

Therefore, $\text{mes}(I_p \cap (t_1, t_n]) = \text{mes}(I_p)$ is bounded from above by the right hand side of (10). We thus can take

$$B = \frac{\pi/2 - \theta_0}{\sin(\theta_0) - \delta} \quad \text{and} \quad C = \pi - \theta_0.$$

Case 3. On the interval $[0, t_{n-1}]$, by Lemma 5.4, $\theta < \theta_0 + \delta$. Since $w(l_{n-1}) < \pi/2$, $\theta^+(t_{n-1}) < \theta_0 + \delta + \pi/2$. On the interval $(t_{n-1}, t_n]$, θ does not increase, and we obtain $\theta \leq \theta_0 + \delta + \pi/2 < \pi$. Therefore, θ is bounded by $\theta_0 + \delta + \pi/2$ for all $t \in [0, P]$.

If $\theta(t) > \theta_0 + \delta$, then

$$\theta_0 + \delta < \theta(t) < \theta_0 + \delta + \pi/2,$$

and, since $\sin(\theta)$ is convex on $[0, \pi]$,

$$\theta'(t) \leq -\sin(\theta(t)) < -\min\{\sin(\theta_0 + \delta + \pi/2), \sin(\theta_0 + \delta)\} < 0.$$

Then

$$\begin{aligned} \theta(t) &= \theta^+(t_{n-1}) + \int_{t_{n-1}}^t \theta'(u) du \\ &< \theta^+(t_{n-1}) - (t - t_{n-1}) \min\{\sin(\theta_0 + \delta + \pi/2), \sin(\theta_0 + \delta)\}. \end{aligned}$$

Therefore, if $\delta + \theta_0 < \theta(t)$, then

$$(11) \quad t - t_{n-1} < \frac{\pi/2}{\min\{\sin(\theta_0 + \delta + \pi/2), \sin(\theta_0 + \delta)\}}.$$

Thus $\text{mes}(I_p \cap (t_{n-1}, t_n]) = \text{mes}(I_p)$ is bounded by the right hand side of (11).

Case 4. This case is basically the combination of Case 2 and Case 3. In $[0, t_{n-1}]$, by an argument similar to that of Case 2, θ is bounded by $\pi - \theta_0$. In addition, the inequalities (7), (8) hold, and $\text{mes}(I_p \cap [t_0, t_{t-1}])$ is bounded from above by the right hand side of (10). Therefore, if $\theta(t_{n-1}) > \theta_0 + \delta$,

$$\theta(t_{n-1}) \leq \theta^+(t_1) - (t_{n-1} - t_1)(\sin \theta_0 - \delta) + \delta,$$

by Hypothesis (iii) and the relation between δ, θ_0 and D ,

$$< \pi/2 + D(\delta - \sin(\theta_0)) + \delta < \pi/2.$$

Let $M = \max\{\theta_0 + \delta, \pi/2 + D(\delta - \sin(\theta_0)) + \delta\} < \pi/2$. Then $\theta^+(t_{n-1}) \leq M + \pi/2$. On $(t_{n-1}, t_n]$, since θ does not increase, $\theta \leq M + \pi/2 < \pi$.

If $\theta(t) > \theta_0 + \delta$, then

$$\begin{aligned} \theta'(t) &= -\sin(\theta(t)) \leq -\min\{\sin(M + \pi/2), \sin(\theta_0 + \delta)\} < 0, \\ \theta_0 + \delta &< \theta(t) < \theta^+(t_{n-1}) - (t - t_{n-1}) \min\{\sin(M + \pi/2), \sin(\theta_0 + \delta)\}, \\ (12) \quad \text{and} \quad t - t_{n-1} &< \frac{M + \pi/2 - (\theta_0 + \delta)}{\min\{\sin(M + \pi/2), \sin(\theta_0 + \delta)\}}. \end{aligned}$$

Therefore, $\text{mes}(I_p \cap (t_{n-1}, t_n])$ is bounded by the right hand side of (12). Hence, $\text{mes}(I_p)$ is bounded by the sum of the right hand sides of (10) and (12), and θ is bounded by $\max\{\pi - \theta_0, M + \pi/2\} < \pi$.

In each Case 1 – Case 4, we have found upper bounds of θ_p and $\text{mes}(I_p)$ that do not depend on the choices for p , L and L' . The proposition immediately follows. \square

(Continue the proof of Theorem 5.2.) Let $x = p(0)$ and $y = p(P)$, which are the end points of the geodesic segment $p : [0, P] \rightarrow \mathbb{H}^2$. Then $\text{dist}_{\mathbb{H}^2}(x, y) = P$. We have

$$\begin{aligned} \text{dist}(\beta(x), \beta(y)) &= d_p(P) = \int_0^P d'_p(t) dt \\ &= \int_{[0, P] \setminus I_p} \cos(\theta(t)) dt + \int_{I_p} \cos(\theta(t)) dt. \end{aligned}$$

When $t \notin I_p$, $\cos(\theta(t)) \geq \cos(\theta_0 + \delta)$, while by Proposition 5.3, $\text{mes}(I_p)$ is bounded by C . Therefore,

$$\begin{aligned} \text{dist}(\beta(x), \beta(y)) &> \cos(\theta_0 + \delta)(P - C) - C \\ &= \cos(\theta_0 + \delta)\text{dist}(x, y) - C(1 + \cos(\theta_0 + \delta)). \end{aligned}$$

On the other hand,

$$\text{dist}(\beta(x), \beta(y)) = \int_0^P \cos(\theta(t)) dt \leq \int_0^P dt = \text{dist}(x, y).$$

Recall that p is an arbitrary geodesic in \mathbb{H}^2 . Accordingly, x and y are arbitrary points in \mathbb{H}^2 . Therefore, letting

$$S = \frac{1}{\cos(\theta_0 + \delta)} \quad \text{and} \quad T = C(1 + \cos(\theta_0 + \delta)),$$

we conclude that β is an (S, T) -quasiisometric embedding.

We claim that β is injective. If $\beta(x) = \beta(y)$ for distinct $x, y \in \mathbb{H}^2$, let p be the geodesic segment connecting x to y . Then $p|_{[t_{n-1}, t_n]}$ is the geodesic segment connecting $p(t_{n-1})$ to x . Then, for $t \in (t_{n-1}, t_n)$, $\theta(t) = \angle \gamma(x) \gamma(t) \gamma(t_{n-1}) = \pi$. This contracts the upper bound $C < \pi$ of θ . Therefore, β is injective.

General Case. For the rest of the proof, we deal with the case that L does *not* consist of isolated leaves. Take a sequence of measured laminations on \mathbb{H}^2 , $(L_i = (\lambda_i, \mu_i))$, such that each L_i consists of finitely many leaves and (L_i) limits to L in Thurston's topology. Let $\beta_{L_i} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ denote the bending map induced by L_i . Then β_{L_i} converges to β_L uniformly on compacts. (See [EM] for the construction of (L_i) and the convergence of (β_{L_i}) .) For an arbitrary geodesic segment $p : [0, P] \rightarrow \mathbb{H}^2$, let $\gamma_i = \beta_{L_i} \circ p : [0, P] \rightarrow \mathbb{H}^3$. By the Discrete Case, we have

$$\frac{1}{S} \text{dist}(x, y) - T < \text{dist}(\gamma_i(0), \gamma_i(P)) = \text{dist}(\beta_{L_i}(x), \beta_{L_i}(y)) \leq \text{dist}(x, y)$$

for sufficiently large i . In addition, by the convergence of β_{L_i} to β_L , $\text{dist}(\gamma_i(0), \gamma_i(P))$ limits to $\text{dist}(\gamma(0), \gamma(P))$ as $i \rightarrow \infty$. Then

$$\frac{1}{S} \text{dist}(x, y) - T \leq \text{dist}(\gamma(0), \gamma(P)) = \text{dist}(\beta_L(x), \beta_L(y)) \leq \text{dist}(x, y).$$

Therefore, β_L is an (S, T) -quasiisometric embedding.

We claim that β_L is injective. Suppose that β_L is not injective. Then there exist distinct $x, y \in \mathbb{H}^2$ with $\beta_L(x) = \beta_L(y)$. Let $p : [0, P] \rightarrow \mathbb{H}^2$ be the geodesic segment from x to y . There are only countably many leaves of L with positive weight (in particular, near y). Note that the transversal measure restricted to the other leaves of L is non-atomic. Therefore, $\mu(\overline{p(t)p(P)})$ limits to 0 as $t \nearrow P$ (recall that, if $p(P) = y$ is contained in a leaf of L with positive weight, then the weight does not contribute to the transversal measure $\mu(\overline{p(t)p(P)})$ by our convention). Thus, for every small $\epsilon > 0$, there exists $z \in \overline{xy} \setminus \{x, y\} = p((0, P))$ such that $\mu(\overline{zy}) < \epsilon$. Therefore, for every $\epsilon > 0$, there exist $0 < t_1 < t_2 < t_3 < P$ such that, letting $z_i = p(t_i)$ ($i = 1, 2, 3$), $\mu(\overline{z_1 y}) < \epsilon$, $\text{dist}(z_3, y) < \epsilon$ and

$$(13) \quad \frac{\text{dist}(z_3, y)}{\text{dist}(z_2, z_3)} < \epsilon.$$

For sufficiently large i , $\mu_i(\overline{z_2 z_3}) < \epsilon$ by the convergence of μ_i to μ . Recall that $\gamma_i = \beta_{L_i} \circ p$, where β_{L_i} is the bending map induced by L_i . Let $t_i^- \in [0, t_2)$, $t_i^+ \in (t_2, P]$ be the non-smooth points of γ_i closest to t_2 in $[0, t_2)$ and $(t_2, P]$, respectively. The convergence of (L_i) implies that, for sufficiently large i , ϵ bounds from above the sum of the external angles at the non-smooth points of γ_i on (t_2, t_3) . In the following, we always assume that ϵ is sufficiently small. Since γ_i is “almost straight” on (t_2, t_3) , for every $\eta > 0$, $\angle \gamma_i(t_3) \gamma_i(t_2) \gamma_i(t_i^+) < \eta$ provided that i is sufficiently large. (See Figure 9.) For sufficiently large i , we also have $\angle \gamma_i(t_3) \gamma_i(t_2) \gamma_i(P) < \eta$ by (13), and $\angle \gamma_i(0) \gamma_i(t_2) \gamma_i(P) < \eta$ since

$\gamma_i(0)$ and $\gamma_i(P)$ are close to each other and far from $\gamma_i(t_2)$. If γ_i is not smooth at t_2 , the external angle of γ_i at t_2 is $\pi - \angle \gamma_i(t_i^-) \gamma_i(t_2) \gamma_i(t_i^+)$, and, since $\mu(\overline{z_1 y}) < \epsilon$, the external angle is bounded from above by ϵ . Let θ_i be the angle function of γ_i . Then, by the triangle inequality, $\theta_i(t_2) = \angle \gamma_i(0) \gamma_i(t_2) \gamma_i(t_i^-) > \pi - (3\eta + \epsilon)$ (see Figure 9). Therefore, by taking sufficiently small ϵ and η , $\theta_i(t_2) > C$ for large i , where $0 < C < \pi$ is the constant obtained from Proposition 5.3. Contradictorily, by Proposition 5.3, we can show $\theta_i < C$ for large i . \square

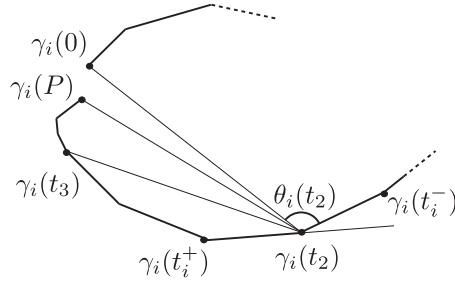


FIGURE 9.

Assume that L satisfies the assumptions of Theorem 5.1 or Theorem 5.2. By these theorems, $\beta = \beta_L : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is an injective quasiisometric embedding, and, hence, it extends continuously to a homeomorphism $\partial\beta$ from $\partial\mathbb{H}^2$ onto a simple loop on $\partial\mathbb{H}^3 \cong \hat{\mathbb{C}}$ (see [Gh], [Gr]). Therefore, $\hat{\mathbb{C}} \setminus \text{Im}(\partial\beta)$ consists of two simply connected regions.

Corollary 5.6. *Under the assumption of Theorem 5.1 or Theorem 5.2, the projective structure $C(L)$ on \mathbb{D}^2 corresponding to L is admissible.*

Proof. Since β is an injective continuous quasiisometric embedding, $\text{Im}(\beta)$ is a proper surface embedded in \mathbb{H}^3 . Therefore, $\text{Im}(\beta)$ separates \mathbb{H}^3 into two components (the Jordan-Brouwer Separation Theorem). Since $\text{Im}(\beta)$ is locally convex, one of the components of $\mathbb{H}^3 \setminus \text{Im}(\beta)$ is convex (see [CEG, I.1.3]).

The concave component of $\mathbb{H}^3 \setminus \text{Im}(\beta)$ is cobounded by $\text{Im}(\beta)$ and a topological closed disk D contained in $\hat{\mathbb{C}}$. Then $\text{Im}(\beta) = \partial \text{Conv}(\hat{\mathbb{C}} \setminus D)$. Since L does not contain leaves with weight $\geq \pi$, L is the canonical bending lamination on $\partial \text{Conv}(\hat{\mathbb{C}} \setminus D)$. Therefore, (\mathbb{H}^2, L) is Thurston's coordinates for the projective structure on \mathring{D} ; Hence $C(L)$ is admissible. \square

6. THE EXISTENCE OF ADMISSIBLE LOOPS

Let S be a closed orientable surface of genus at least 2. Let $C = (f, \rho)$ be a projective structure on S . Express C in Thurston's coordinates as (τ, L) , where τ is a marked hyperbolic structure on S and $L = (\lambda, \mu)$ is a measured lamination on (S, τ) . Let us recall other related notions from §3: Let $\tilde{L} = (\tilde{\lambda}, \tilde{\mu})$ be the total lift of L to \mathbb{H}^2 , and let $\beta_{\tilde{L}} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by \tilde{L} . Let $\kappa : (S, C) \rightarrow (S, \tau)$ be the collapsing map, and let $\tilde{\kappa} : (\tilde{S}, \tilde{C}) \rightarrow (\mathbb{H}^2, \tilde{L})$ be the lift of κ to a map between the universal covers of (S, C) and (S, τ) . Let $L' = (\lambda', \mu')$ be the canonical (topological) measured lamination on (S, C) corresponding to L via κ (§3.8). Let \tilde{L}' be the total lift of L' to (\tilde{S}, \tilde{C}) .

Since L decomposes into minimal measured laminations, we can set

$$L = (P_1 \sqcup P_2 \sqcup \dots \sqcup P_m) \sqcup (M^1 \sqcup M^2 \sqcup \dots \sqcup M^n),$$

where P_h , $h = 1, \dots, m$, are the periodic minimal sublaminations of L and $M^j = (\nu^j, \omega^j)$, $j = 1, \dots, n$, are the irrational minimal sublaminations of L . Let $p_h = |P_h|$ denote the periodic leaf supporting P_h , and let p denote the periodic part of $|\lambda|$,

$$p_1 \sqcup p_2 \sqcup \dots \sqcup p_m.$$

Let $M = (\nu, \omega)$ denote the irrational part of L ,

$$M^1 \sqcup M^2 \sqcup \dots \sqcup M^n.$$

For each $j \in \{1, 2, \dots, n\}$, let $(l_i^j)_{i=1}^\infty$ be the sequence of simple geodesic loops on (S, τ) that approximates $|\nu^j|$, constructed in §3.5. By Lemma 3.3, we have $\lim_{i \rightarrow \infty} \omega^j(l_i^j) = 0$. Since $|\nu^j|$ is an isolated subset of $|\lambda|$, we can assume that l_i^j does not intersect $|\lambda \setminus \nu^j|$. Therefore, $\lim_{i \rightarrow \infty} \mu(l_i^j) = 0$. Let $l_i = l_i^1 \sqcup \dots \sqcup l_i^n$, so that $\lim_{i \rightarrow \infty} l_i = |\nu|$.

Recall that, for each h , $\kappa^{-1}(p_h)$ is a flat foliated cylinder of height $w(p_h)$ in (S, C, L') , where $w(p_h)$ is the weight of p_h . The foliation on $\kappa^{-1}(p_h)$ consists of admissible loops that are homeomorphic to p_h via κ (§3.8).

Recall also that κ is a C^1 -diffeomorphism on $S \setminus \kappa^{-1}(p)$. Therefore, if l is an essential simple loop on (S, τ) disjoint from $|\lambda|$, then $\kappa^{-1}(l)$ is an essential simple loop on (S, C) disjoint from $|\lambda'|$. We shall see that $\kappa^{-1}(l)$ is also admissible. Let P be the component of $(S, \tau) \setminus |L|$ containing l , and let \tilde{l} and \tilde{P} be the lifts of l and P , respectively. We can assume that $\tilde{l} \subset \tilde{P}$ and that $\pi_1(l)$ acts on \tilde{l} freely and properly discontinuously as an infinite cyclic subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by

a hyperbolic element. Let γ_l be the homotopy class of l that generates $\pi_1(l)$. Since \tilde{l} is a quasigeodesic in \mathbb{H}^2 and $\beta_{\tilde{L}}$ is an isometry on \tilde{P} , $\beta_{\tilde{L}}(\tilde{l})$ is a quasigeodesic in \mathbb{H}^3 . Then $\pi_1(l)$ acts on $\beta_{\tilde{L}}(\tilde{l})$ freely and properly discontinuously via ρ , and this action fixes the distinct end points of $\beta_{\tilde{L}}(\tilde{l})$ on $\hat{\mathbb{C}}$. Therefore, $\rho(\gamma_l)$ is a loxodromic element of $\mathrm{PSL}(2, \mathbb{C})$. The curve $\tilde{\kappa}^{-1}(\tilde{l})$ is a lift of $\kappa^{-1}(l)$ to (\tilde{S}, \tilde{C}) . Then $\tilde{\kappa}^{-1}(\tilde{l})$ is contained in $\tilde{\kappa}^{-1}(\tilde{P})$, which is a component of $(\tilde{S}, \tilde{C}) \setminus |\tilde{L}'|$. Since $cl(\tilde{\kappa}^{-1}(\tilde{P}))$ is the core of a maximal ball, $\tilde{\kappa}^{-1}(\tilde{l})$ is contained in a maximal ball. Thus, f is an embedding on $\tilde{\kappa}^{-1}(\tilde{l})$. Hence, $\kappa^{-1}(l)$ is admissible.

We have $\mu(p_h) = 0$ and $\mu(l) = 0$. For each i, j , let $l_i^j = \kappa^{-1}(l_i^j)$, which is a simple loop on (S, C) . The fact that $\lim_{i \rightarrow \infty} \mu(l_i^j) = 0$ suggests the following proposition:

Proposition 6.1. *For each $j \in \{1, 2, \dots, n\}$, l_i^j is admissible, provided that i is sufficiently large.*

Proof. Let \tilde{l}_i^j be a lift of l_i^j to \mathbb{H}^2 . Consider the measured lamination $I(\tilde{L}, \tilde{l}_i^j)$. Since l_i^j is disjoint from $|\lambda \setminus \nu^j|$, $I(\tilde{L}, \tilde{l}_i^j) = I(\tilde{M}^j, \tilde{l}_i^j)$, where \tilde{M}^j is the total lift of M^j . Choose $\delta > 0$ as in Theorem 5.1. Applying Proposition 4.2 with $I(\tilde{L}, \tilde{l}_i^j) = I(\tilde{M}^j, \tilde{l}_i^j)$, we have $\|I(\tilde{L}, \tilde{l}_i^j)\| < \delta$ for all large i . By Theorem 5.1, for sufficiently large i , the bending map $\beta_{I(\tilde{L}, \tilde{l}_i^j)}$ induced by $I(\tilde{L}, \tilde{l}_i^j)$ is an injective quasiisometric embedding, and it continuously extends to an embedding of $\partial\mathbb{H}^2$.

Let γ_i^j be the homotopy class of l_i^j that acts on \mathbb{H}^2 as a hyperbolic element of $\mathrm{PSL}(2, \mathbb{R})$ preserving \tilde{l}_i^j . The extension of $\beta_{I(\tilde{L}, \tilde{l}_i^j)}$ homeomorphically takes the limit set of $\langle \gamma_i^j \rangle$ to the limit set of $\langle \rho(\gamma_i^j) \rangle$. Thus $\rho(\gamma_i^j)$ is loxodromic.

By Corollary 5.6, the projective structure $C(I(\tilde{L}, \tilde{l}_i^j))$ on \mathbb{D}^2 corresponding to $I(\tilde{L}, \tilde{l}_i^j)$ is admissible for sufficiently large i . Then the developing map f_I of $C(I(\tilde{L}, \tilde{l}_i^j))$ is an embedding. By Lemma 3.9, there exists a homeomorphism $\phi : \kappa_I^{-1}(\tilde{l}_i^j) \rightarrow \kappa^{-1}(\tilde{l}_i^j)$ such that $f_I = f \circ \phi$ on $\kappa_I^{-1}(\tilde{l}_i^j)$, where $\kappa_I : \mathbb{D}^2 \rightarrow \mathbb{H}^2$ is the collapsing map for $C(I(\tilde{L}, \tilde{l}_i^j))$. Since f_I is an embedding, $f = f_I \circ \phi^{-1}$ restricted to \tilde{l}_i^j is an embedding. \square

We thus obtain an admissible loop from every minimal sublamination of L and every complementary region of $|\lambda|$ that is not topologically an open disk. Therefore,

Theorem 6.2. *For every projective structure C on S , there exists an admissible loop on (S, C) .*

Remark: Equivalently, we can state that every projective structure on S admits a grafting operation.

7. ADMISSIBLE DECOMPOSITION

We carry over our notation from the previous section. We have shown that l_i^j and p_h correspond to admissible loops on S through κ , provided that i is sufficiently large. Their union

$$l_i \sqcup p = (l_i^1 \sqcup \dots \sqcup l_i^n) \sqcup (p_1 \sqcup \dots \sqcup p_m)$$

is a multi-loop on (S, τ) . In this section, we show that $l_i \sqcup p$ decomposes (S, C) into admissible subsurfaces.

Theorem 7.1 (Admissible Decomposition). *Let C be a projective structure on a closed orientable surface S of genus at least 2. Then there exists a decomposition of S into cylinders and compact subsurfaces of negative Euler characteristic, such that the restriction of C to each cylinder is an integral flat structure and the restriction to each subsurface of negative Euler characteristic is an admissible projective structure.*

Note that every flat cylinder of height less than 2π is admissible. Therefore, every integral flat cylinder can be further decomposed into admissible flat cylinders, if we wish. Moreover, by further decomposing each surface of negative Euler characteristic into pairs of pants, if necessary, we immediately obtain:

Corollary 7.2. *There exists a decomposition of S into pairs of pants and cylinders such that the restriction of C to each cylinder is an integral flat structure and the restriction to each pair of pants is an admissible structure.*

Let l be a geodesic lamination on a complete hyperbolic surface F . Let $\mathcal{NT}(l)$ denote the collection of all geodesic segments of length less than one on F that do *not* transversally intersect any leaves of l . Then, a geodesic segment s connecting x and y on F is an element of $\mathcal{NT}(l)$ if and only if either $s \subset |l|$ or $(s \setminus \{x, y\}) \cap |l| = \emptyset$.

Lemma 7.3. *For every $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that, if $i > i_0$, then $\omega(s) < \epsilon$ for all $s \in \mathcal{NT}(l_i)$.*

Proof. We claim that, for every $x \in (S, \tau)$, there exist a neighborhood U_x of x and $i_x \in \mathbb{N}$ such that, if $i > i_x$, then $\omega(s) < \epsilon$ for every $s \in \mathcal{NT}(l_i)$ with $s \cap U_x \neq \emptyset$. This would imply the Lemma, since S is compact. Let $\tilde{M} = (\tilde{\nu}, \tilde{\omega})$ and \tilde{l}_i denote the total lifts of M and l_i to \mathbb{H}^2 , respectively. Choose a lift \tilde{x} of x to \mathbb{H}^2 . Through the covering map from \mathbb{H}^2 to (S, τ) , the above claim is equivalent to the following: There exist a neighborhood $U_{\tilde{x}}$ of \tilde{x} and $i_{\tilde{x}} \in \mathbb{N}$ such that, if $i > i_{\tilde{x}}$, then $\tilde{\omega}(s) < \epsilon$ for every $s \in \mathcal{NT}(\tilde{l}_i)$ such that $s \cap U_{\tilde{x}} \neq \emptyset$.

Case 0. Suppose first that $x \notin |\nu|$; Then, $\tilde{x} \notin |\tilde{\nu}|$. Let P be the component of $\mathbb{H}^2 \setminus |\tilde{\nu}|$ that contains \tilde{x} . Then P is an open convex region bounded by some leaves of $\tilde{\nu}$. Clearly, only finitely many such boundary leaves intersect $D_2(\tilde{x})$. Let m_1, m_2, \dots, m_k denote these intersecting leaves. Let $m'_h = D_2(\tilde{x}) \cap m_h$ for all $h \in \{1, 2, \dots, k\}$. (See Figure 10.) Recall that there is a projection $\mathcal{P} : \mathbb{H}^2 \rightarrow T$, where T is the tree dual to \tilde{M} (§3.3). By Lemma 3.1, for every $\epsilon > 0$ and every $h \in \{1, 2, \dots, k\}$, there exists a flow box neighborhood V_h of m'_h that projects to a geodesic segment of length less than $\epsilon/2$ in T . Let s_h denote the geodesic segment $\mathcal{P}(V_h)$. Since m_h is a boundary geodesic of P , the leaves of \tilde{M} contained in P do not accumulate to m_h . Therefore, by the construction of V_h , we can assume that $\mathcal{P}(m'_h) = \mathcal{P}(m_h) = \mathcal{P}(P)$ is an end point of s_h . Therefore, $\cup_h V_h$ projects to $\vee_h s_h \subset T$, the one point union of s_h that identifies the end points $\mathcal{P}(m_h)$ of s_h . Then $(\cup_h V_h) \cup (P \cap D_2(\tilde{x}))$ also projects onto $\vee_h s_h$, and the diameter of $\vee_h s_h$ is less than ϵ .

Since the sequence (\tilde{l}_i) approximates $\tilde{\nu}$, each m_h is approximated by a sequence $(n_{h,i})_i$ such that $n_{h,i}$ is a leaf of \tilde{l}_i . For the rest of Case 0, we always assume that i is sufficiently large. For each i , let P_i be the open convex region bounded by $\bigsqcup_h n_{h,i}$. Then (P_i) limits to P as i goes to infinity. Hence, $\tilde{x} \in P_i \cap D_2(\tilde{x}) \subset (\cup_h V_h) \cup (P \cap D_2(\tilde{x}))$. Take an open neighborhood $U_{\tilde{x}}$ of \tilde{x} such that the closure of $U_{\tilde{x}}$ is contained in the interior of $P \cap D_1(\tilde{x})$. Then $U_{\tilde{x}}$ is contained in P_i . Since $\partial P_i \subset |\tilde{l}_i|$, every $s \in \mathcal{NT}(\tilde{l}_i)$ with $s \cap U_{\tilde{x}} \neq \emptyset$ is contained in $cl(P_i) \cap \mathring{D}_2(\tilde{x})$, and therefore s is contained in $(\cup_h V_h) \cup (P \cap D_2(\tilde{x}))$. Hence $\mathcal{P}(s) \subset \vee_h s_h$ and $\tilde{\omega}(s) < \epsilon$.

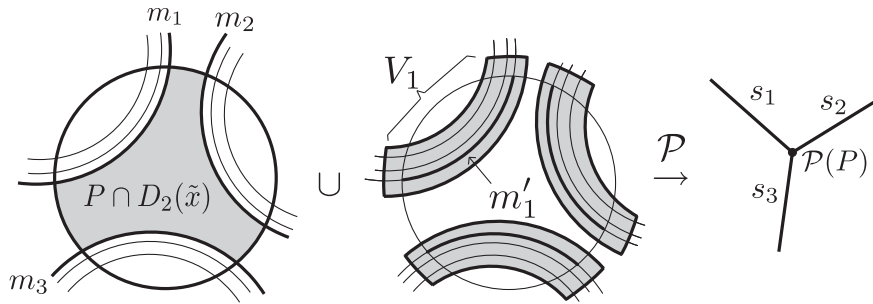


FIGURE 10. Case 0.

Next, suppose $x \in |\nu|$; then, $\tilde{x} \in |\tilde{\nu}|$. Let $l \in \tilde{\nu}$ be the leaf containing \tilde{x} , and let $l' = l \cap D_2(\tilde{x})$. There are two ways that $\tilde{\nu}$ can accumulate to l :

Case 1. the leaves of $\tilde{\nu}$ accumulate from only one side of l , or

Case 2. the leaves of $\tilde{\nu}$ accumulate from both sides of l .

In both cases, by Lemma 3.1, there exists a flow box neighborhood V of l' with height less than $\epsilon/2$.

In Case 1, l is a boundary geodesic of a component P' of $\mathbb{H}^2 \setminus |\tilde{\nu}|$. Let m_1, m_2, \dots, m_k be the boundary geodesics of P' that intersect $D_2(\tilde{x})$. We can assume that $m_1 = l$. Then l bounds a half plane disjoint from P' , and the leaves of $\tilde{\nu}$ contained in this half plane accumulate to l . Consider a leaf m of $\tilde{\nu}$ in this half plane, such that $m \cap D_2(\tilde{x}) \subset V$. Let P be the open convex region in \mathbb{H}^2 bounded by m and m_2, m_3, \dots, m_h , which contains \tilde{x} . Then $\mathcal{P}(P)$ is a geodesic segment of length less than $\epsilon/2$. Take an open neighborhood $U_{\tilde{x}}$ of \tilde{x} such that the closure of $U_{\tilde{x}}$ is contained in the interior of $P \cap D_1(\tilde{x})$. The same argument as Case 0 shows that $U_{\tilde{x}}$ satisfies the desired property. (See the left picture in Figure 11.)

In Case 2, in each component of $\mathbb{H}^2 \setminus l$, consider a leaf m_i , $i = 1, 2$, of $\tilde{\nu}$ close to l . Let P be the open convex region in \mathbb{H}^2 bounded by m_1 and m_2 , which contains l . By taking m_1 and m_2 sufficiently close to l , we can assume that $P \cap D_2(\tilde{x})$ is contained in V . Take an open neighborhood $U_{\tilde{x}}$ of \tilde{x} such that the closure of $U_{\tilde{x}}$ is contained in the interior of $P \cap D_1(\tilde{x})$. Again, as in Case 0, we see that $U_{\tilde{x}}$ satisfies the desired property. (See the right picture in Figure 11.) \square

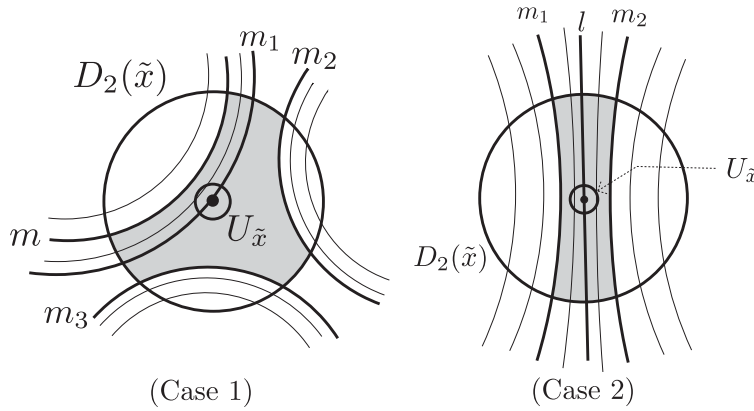


FIGURE 11. In each picture, $P \cap D_2(\tilde{x})$ is shaded.

Let Q be a component of $S \setminus l_i$, and let \tilde{Q} be a lift of Q to \mathbb{H}^2 . Note that, if we take a different lift \tilde{Q} , then $I(\tilde{M}, \tilde{Q})$ changes only by an element of $\pi_1(S)$. In particular, $\|I(\tilde{M}, \tilde{Q})\|$ does not depend on the choice of \tilde{Q} .

Proposition 7.4. *For every $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that, if $i > i_0$, then $\|I(\tilde{M}, \tilde{Q})\| < \epsilon$ for every component Q of $S \setminus l_i$.*

Proof. For every Q , its lift \tilde{Q} is an open convex region bounded by some leaves of \tilde{l}_i . Therefore, $\mathbb{H}^2 \setminus \tilde{Q}$ consists of closed half planes bounded by these leaves. We can assume that \tilde{Q} is $\pi_1(Q)$ -invariant. Let k be the number of the boundary components of Q . The $\pi_1(Q)$ -action permutes the complementary half planes, and this action on the half planes has exactly k orbits. Let H_1, H_2, \dots, H_k be the representatives of the orbits. Since l consists of n disjoint simple loops on S , $k \leq 2n$.

Let H be a component of $\mathbb{H}^2 \setminus \tilde{Q}$. Note that ∂H is transversal to \tilde{M} . Then a leaf l of $I(\tilde{M}, \tilde{Q})$ intersects H if and only if l intersects ∂H . Therefore, $I(I(\tilde{M}, \tilde{Q}), H) = I(\tilde{M}, \partial H)$. By the covering map from \mathbb{H}^2 to (S, τ) , ∂H covers some l_i^j . Recalling that \tilde{l}_i^j is a lift of l_i^j to \mathbb{H}^2 , $I(\tilde{M}, \partial H) \cong I(\tilde{M}, \tilde{l}_i^j)$. Since $I(\tilde{M}, \tilde{Q})$ and $I(I(\tilde{M}, \tilde{Q}), H)$ coincide on H ,

$$\|I(\tilde{M}, \tilde{Q})|_H\| = \|I(I(\tilde{M}, \tilde{Q}), H)|_H\| \leq \|I(I(\tilde{M}, \tilde{Q}), H)\| = \|I(\tilde{M}, \tilde{l}_i^j)\|.$$

Since \tilde{Q} is convex, for every geodesic segment s in \mathbb{H}^2 , $\tilde{Q} \cap s$ is either empty or a geodesic segment. In addition, $s \cap (\mathbb{H}^2 \setminus \tilde{Q})$ consists of at most 2 geodesic segments, each of which is contained in a component of $\mathbb{H}^2 \setminus \tilde{Q}$ (see Figure 12). Therefore, by the definition of the norm,

$$\begin{aligned} \|I(\tilde{M}, \tilde{Q})\| &\leq \|I(\tilde{M}, \tilde{Q})|_{\tilde{Q}}\| + 2 \max\{\|I(\tilde{M}, \tilde{Q})|_{H_j}\| \mid j = 1, \dots, m\} \\ (14) \quad &\leq \|I(\tilde{M}, \tilde{Q})|_{\tilde{Q}}\| + 2 \max\{\|I(\tilde{M}, \tilde{l}_i^j)\| \mid j = 1, \dots, m\}. \end{aligned}$$

By Lemma 7.3, for every $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that, if $i > i_0$, then $\|I(\tilde{M}, \tilde{Q})|_{\tilde{Q}}\| < \epsilon$ for every component Q of $S \setminus l_i$. By Proposition 4.2, for every j , $\|I(\tilde{M}, \tilde{l}_i^j)\| \rightarrow 0$ as $i \rightarrow \infty$. Therefore, for every $\epsilon > 0$, if i is sufficiently large, then (14) is bounded from above by ϵ for every Q . \square

Proof. (Theorem 7.1) Recall that m is the number of the periodic leaves of L . For each $k \in \{1, 2, \dots, m\}$, let $A_k = \kappa^{-1}(p_k) \subset (S, C)$, which is a flat cylinder of height $w(p_k)$, and let $A = \bigsqcup_{k=1}^m A_k$. Since $S \setminus A$ is homeomorphic to $S \setminus p$ by κ , $\kappa^{-1}(l_i^j)$ is a simple closed loop on (S, C) for each sufficiently large $i \in \mathbb{N}$ and for each $j \in \{1, 2, \dots, m\}$. Let $l_i^j = \kappa^{-1}(l_i^j)$ and $l'_i = \bigsqcup_j l_i^j$. Then κ restricts to a homeomorphism from $S \setminus (A \bigsqcup l'_i)$ to $S \setminus (p \bigsqcup l_i)$. In particular, the components of $S \setminus (A \bigsqcup l'_i)$ and $S \setminus (p \bigsqcup l_i)$ correspond bijectively. Note that, since p_h 's and l_i^j 's are not pairwise homotopic (when i is fixed), each component of $S \setminus (A \bigsqcup l'_i)$

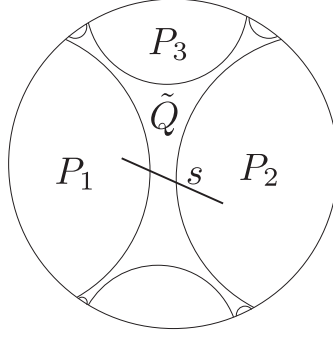


FIGURE 12.

and $S \setminus (p \sqcup l_i)$ is (the interior of) a compact surface with boundary of negative Euler characteristic.

Let $S'_i = S \setminus (A \sqcup l'_i)$. First, we shall show that, when i is large, every component R' of S'_i is admissible. Let $R = \kappa(R')$, which is a component of $(S, \tau) \setminus (l_i \sqcup p)$. Let \tilde{R} and \tilde{R}' be the corresponding lifts of R and R' to the universal covers, (\tilde{S}, \tilde{C}) and $(\mathbb{H}^2, \tilde{L})$, respectively, so that $\tilde{\kappa}(\tilde{R}') = \tilde{R}$. Choose $\delta > 0$ that satisfies Theorem 5.1. Let $I_R = I(\tilde{M}, \tilde{R})$. Then, since $R \cap p = \emptyset$, $I_R = I(\tilde{L}, \tilde{R})$. Note that R is contained in a component of $S \setminus l_i$. Therefore, by Proposition 7.4, $\|I_R\| < \delta$ for every component R' of S'_i , provided that i is sufficiently large. Hence, by Corollary 5.6, the projective structure $C(I_R)$ on $\mathring{\mathbb{D}}^2$ is admissible. Since $I_R = I(\tilde{L}, \tilde{R})$ and \tilde{R} is a convex subset of \mathbb{H}^2 bounded by geodesics, by Lemma 3.9, $C(I_R)$ embeds into $C(\tilde{L})$. Each boundary leaf of R does not transversally intersect a leaf of L with positive weight and R is not a geodesic. Therefore, by Corollary 3.10, $\mathcal{R}(\tilde{C}, \tilde{R}') = \mathcal{R}(C(I_R), \kappa_{I_R}^{-1}(\tilde{R})) \subset C(I_R)$, where κ_{I_R} is the collapsing map for $C(I_R)$. Since $C(I_R)$ is admissible, $\mathcal{R}(\tilde{C}, \tilde{R}')$ is also admissible.

Through its action, $\pi_1(R) \cong \pi_1(R')$ is regarded as a Schottky group in $\mathrm{PSL}(2, \mathbb{R})$. Let $\beta_{I_R} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by I_R . Then, by Theorem 5.1, β_{I_R} is an injective quasiisometric embedding, and it extends to an equivariant embedding of $\partial\mathbb{H}^2$ to $\hat{\mathbb{C}}$. In particular, this extension takes the limit set of $\pi_1(R)$ to the limit set of $\rho(\pi_1(R))$ homeomorphically and $\rho|_{\pi_1(R)}$ -equivariantly. Therefore, $\rho|_{\pi_1(R)}$ is an isomorphism onto a Schottky group in $\mathrm{PSL}(2, \mathbb{C})$. Hence, the restriction of C to R' is admissible.

We have given a desired decomposition of (S, C) , except that the flat cylinders A_i are not integral. In what follows, instead of cutting

out the whole A_i from S , we cut out a maximal integral flat cylinder contained in A_i . Taking the union of the maximal integral flat cylinders and l'_i , we shall show that the complementary regions of the union are admissible, which completes the decomposition.

For $x \in \mathbb{R}_{\geq 0}$, let $[x] = \max\{n \in \mathbb{Z}_{\geq 0} \mid 2\pi n \leq x\}$. Besides, let

$$a_k = \frac{w(p_k) - 2\pi[w(p_k)]}{2}$$

for each k . Then $0 \leq a_k < \pi$. Cut each A_k along two admissible loops into three flat cylinders of heights a_k , $2\pi[w(p_k)]$, a_k in this order. The middle cylinder A'_k is integral and the others are not. (See A_1 in Figure 13.) If $w(a_k) < 2\pi$, then A'_k degenerates to an admissible loop in the middle of A_k (see A_2 in Figure 13).

Let $A' = \bigsqcup_{k=1}^m A'_k$ and $S''_i = S \setminus (l'_i \sqcup A')$. Each non-integral flat cylinder obtained above shares exactly one boundary component with a component of $S'_i = S \setminus (l'_i \sqcup A)$. Therefore, each component R' of S'_i is contained in a component P' of S''_i . If a boundary circle l of R' maps to a periodic leaf p_k via κ , then l bounds a flat cylinder of height a_k in P' . If l maps to an approximating loop l'_i by κ , then l is a boundary component of P' . Thus, each component P' of S''_i is the union of a component R' of S'_i and the non-integral flat cylinders sharing a boundary component with R' . In particular, R' is a deformation retract of P' . Letting B_1, B_2, \dots, B_r be these non-integral flat cylinders, set

$$P' = R' \cup (B_1 \cup \dots \cup B_r)$$

(the shaded region in Figure 13). Then $p_{b_1} := \kappa(B_1)$, $p_{b_2} := \kappa(B_2)$, \dots , $p_{b_r} := \kappa(B_r)$ are periodic leaves of L .

We have $\kappa(R') = \kappa(P') =: R$. Let $\tilde{R}' \subset \tilde{P}'$ be lifts of R' and P' to \tilde{S} , respectively. Then $\tilde{\kappa}(\tilde{R}') = \tilde{\kappa}(\tilde{P}') =: \tilde{R}$, which is a lift of R to \mathbb{H}^2 . Let λ_{∂} be the geodesic lamination on \mathbb{H}^2 consisting of the boundary geodesics of \tilde{R} that are lifts of periodic leaves of L . Let $\alpha = \max\{a_k - \pi/2, 0 \mid k = 1, 2, \dots, m\}$; then $0 \leq \alpha < \pi/2$. Assign the weight α to each leaf of λ_{∂} , and obtain a measured lamination L_{∂} on \mathbb{H}^2 . Recall that $I_R = I(\tilde{L}, \tilde{R}) = I(\tilde{M}, \tilde{R})$. Since there are no leaves of L intersecting both \tilde{R} and $|L_{\partial}|$, and L_{∂} consists of isolated leaves of \tilde{L} , therefore, $|L_{\partial}|$ and $|I_R|$ are disjoint. Then let $L_P = I_R \sqcup L_{\partial}$. Each leaf l of L_{∂} is a boundary geodesic of \tilde{R} , and each leaf of I_R intersects \tilde{R} but does not intersect leaves of L_{∂} . Therefore, each l is an outermost leaf of L_P .

We now apply Theorem 5.2 with $L = L_P$ and $L' = L_{\partial}$. We just have checked that $L_{\partial} \subset \partial L$ (Hypothesis (i) of Theorem 5.2). Every leaf of L_{∂} has the weight $\alpha < \pi/2$ (Hypothesis (iv)). Since p_1, \dots, p_m

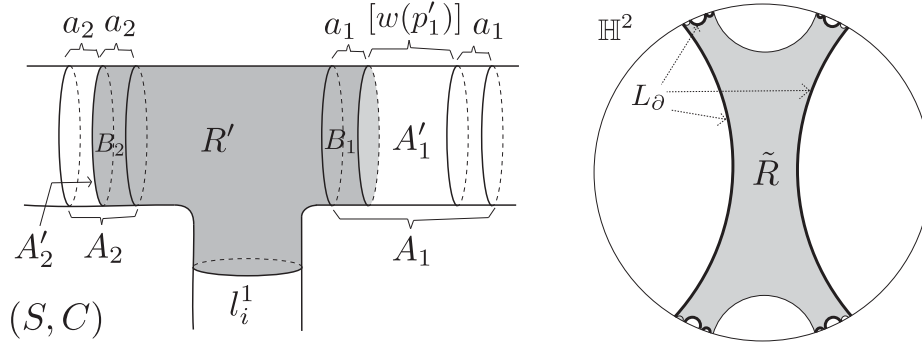


FIGURE 13. In the left picture, the region P' is shaded.
In the right picture, the bold lines are the leaves of L_P .

are disjoint simple loops on S and L_∂ consists of lifts of these loops, there exists $D > 0$ such that every pair of distinct leaves of L_∂ has a distance greater than D (Hypothesis (iii)). Apply Theorem 5.2 to this D , and obtain $\delta, S, T > 0$. Apply Proposition 7.4 with $\epsilon = \delta$, and obtain $i_0 \in \mathbb{N}$. Let Q be the component of $S \setminus l_i$ containing R . Then $I(\tilde{L}, \tilde{R}) = I(\tilde{M}, \tilde{R}) \subset I(\tilde{M}, \tilde{Q})$, where \tilde{Q} is a lift of Q to \mathbb{H}^2 (that contains \tilde{R}). Therefore, by the Proposition 7.4, if $i > i_0$, then $\|I_R\| \leq \|I(\tilde{M}, \tilde{Q})\| < \delta$ (Hypothesis (ii)). By Theorem 5.2, if i is sufficiently large, β_{L_P} is an injective (S, T) -quasiisometric embedding for every component P of S_i'' .

By Corollary 5.6, the projective structure $C(L_P)$ on \mathbb{D}^2 is admissible. Let κ_{I_R} and κ_{L_P} be the collapsing maps associated with $C(I_R)$ and $C(L_P)$, respectively. Since $I_R = I(L_P, \tilde{R}) = I(L, \tilde{R})$, by Corollary 3.10, we have $\mathcal{R}(\tilde{C}, \tilde{R}') = \mathcal{R}(C(I_R), \kappa_{I_R}^{-1}(\tilde{R})) = \mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(\tilde{R}))$. Therefore, $\mathcal{R}(\tilde{C}, \tilde{R}')$ is admissible.

For $h \in \{1, 2, \dots, r\}$, let \tilde{p}_h be a lift of p_{b_h} to \mathbb{H}^2 that bounds \tilde{R} . Let H be the component of $\mathbb{H}^2 \setminus \tilde{R}$ bounded by \tilde{p}_h . Then, since \tilde{R} is open, H is a closed half plane. Note that H is uniquely determined by the choice of h and the choice of the lift of p_{b_h} . Observe that $|L_P| \cap H = \tilde{p}_h$. Then $\mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(H \setminus \tilde{p}_h))$ is a crescent of angle $\pi/2$, and $\mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(\tilde{p}_h))$ is a crescent of angle α . Therefore, $\mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(H))$ is a crescent of angle $\pi/2 + \alpha$, and it is a component of $C(L_P) \setminus \mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(\tilde{R}))$.

Each component of $\mathcal{R}(\tilde{C}, \tilde{P}') \setminus \mathcal{R}(\tilde{C}, \tilde{R}')$ is a lift \tilde{B}_h of some B_h to \tilde{S} . There is a lift \tilde{p}'_h of p_h bounding \tilde{B}_h and \tilde{R}' in (\tilde{S}, \tilde{C}) (Figure 14). The height of \tilde{B}_h is $a_h \leq \pi/2 + \alpha$. By the argument above, when $\mathcal{R}(\tilde{C}, \tilde{R}') =$

$\mathcal{R}(C(L_P), \kappa_{L_P}^{-1}(\tilde{R}))$ is embedded in $C(L_P)$, \tilde{p}'_h bounds a component of $C(L_P) \setminus \mathcal{R}(\tilde{C}, \tilde{R}')$, which is a crescent of angle $\pi/2 + \alpha$. Therefore, this embedding extends to the embedding of $\tilde{B}_h \cup \mathcal{R}(\tilde{C}, \tilde{R}')$ to $C(L_P)$ (see Figure 14). Different components of $\mathcal{R}(\tilde{C}, \tilde{P}') \setminus \mathcal{R}(\tilde{C}, \tilde{R})$ correspond to different \tilde{B}_h . Therefore, the embedding disjointly extends to all components of $\mathcal{R}(\tilde{C}, \tilde{P}') \setminus \mathcal{R}(\tilde{C}, \tilde{R})$ and we obtain an embedding of $\mathcal{R}(\tilde{C}, \tilde{P}')$ into $C(L_P)$. By the construction, the embedding of $\mathcal{R}(\tilde{C}, \tilde{R})$ is $\pi_1(R')$ -equivariant. Since $\mathcal{R}(\tilde{C}, \tilde{P}')$ embeds to $C(L_P)$, and $C(L_P)$ has an injective developing map, therefore $\mathcal{R}(C, P')$ has an injective developing map.

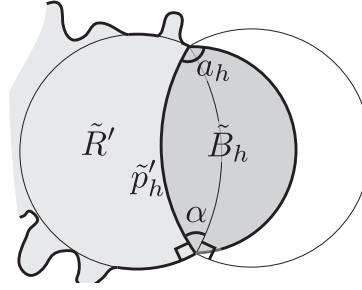


FIGURE 14.

Since R' is a deformation retract of P' , $\pi_1(R')$ equals to $\pi_1(P')$ as subgroups of $\pi_1(S)$. In particular, $\rho|_{\pi_1(P')} = \rho|_{\pi_1(R')}$. We have already seen that $\pi_1(R')$ is isomorphic to a purely loxodromic subgroup of $\text{PSL}(2, \mathbb{C})$ via ρ . Therefore, so is $\rho_{\pi_1(P')}$. Hence, the restriction of C to P' is admissible. \square

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